# *Thermodynamics Based on the Hahn-Banach Theorem: The Clausius Inequality*

M. FEINBERG & R. LAVINE

**Contents**



## 1. Introduction

Although thermodynamics is no longer a young subject, its most elementary results remain under critical scrutiny. That this should be so can perhaps be attributed to the fact that ideas born in the nineteenth century would require twentieth century mathematics for their full and precise expression. Indeed it is only recently that foundations of the subject have been discussed in terms of mathematics that began to appear not long after GIBBS' day.*

For the most part the current discussion remains focused on logical underpinnings of what has come to be called the Clausius-Duhem inequality. Although we regard the work contained here to be a contribution to that discussion, our immediate concern will not be with the Clausius-Duhem inequality in its full generality but rather with the form it takes for cyclic processes. This more limited statement, called the Clausius inequality, is usually deemed to be a logical precursor of the Clausius-Duhem inequality.

If there has been a single impetus for current work on foundations underlying both the Clausius and Clausius-Duhem inequalities, it is probably a paper written almost twenty years ago by COLEMAN and NOLL [CN]. In 1963 they showed how, for elastic materials with heat conduction and viscosity, the Clausius-Duhem inequality could be used systematically in conjunction with the classical conservation laws to deduce restrictions on constitutive relations. The example they set inspired a body of work in which the same methods were applied to draw inferences about constitutive relations in a wide variety of situations. Insofar as it has so extensively been taken up by others, the argument employed by COLEMAN and NOLL seems likely to occupy a permanent place in thermodynamic methodology.

Yet it is no simple matter to trace an unbroken line from premises implicit in modern use of the Clausius-Duhem inequality back to principles of thermodynamics normally held to be fundamental. In particular, it is generally supposed in modern work that even for bodies suffering rapid deformation and heating there can be associated with each material point an entropy density and a thermodynamic temperature, these being functions of the state of the material point, such that the Clausius-Duhem inequality is satisfied for all processes compatible with the standard conservation laws. Although pioneers of thermodynamics from CARNOT to CARATHÉODORY mounted arguments to deduce existence of these functions from various statements of the Second Law, their reasoning is too heavily rooted in consideration of reversible processes to provide an unequivocal basis for crucial suppositions upon which modern use of the Clausius-Duhem inequality depends.

As portrayed in standard textbooks, reversible processes are those which are executed so slowly that a body suffering such a process can be regarded to be in a condition of equilibrium at each instant. With this in mind the more orthodox thermodynamicists hold that, insofar as existence of temperature and entropy density functions is deduced solely from consideration of reversible processes, these functions should have as their domain of definition only those states that might be exhibited in bodies at or near equilibrium. To the extent that modern use of the Clausius-Duhem inequality invokes the existence of entropy and temperature functions defined on states manifested in a far broader class of processes,

* The monograph of TRUESDELL and BHARATHA [TB] suggests how mathematics already available to the nineteenth century thermodynamicists might have lent the subject greater coherency. TRUESDELL's more recent monograph [T1] provides a critical account of the early history of thermodynamics.

its very statement might be regarded to be an extrapolation of conclusions reached by the early thermodynamicists in more restricted circumstances.*

In a sense, then, the great body of modern work takes the Clausius-Duhem inequality *itself*, together with its implicit presumption of entropy and temperature functions, as a new statement of the Second Law somewhat removed from its historical predecessors. Although FOSDICK and SERRIN [FS] have shown the Clausius-Duhem inequality to be consistent with the more classical statements of the Second Law, we may still ask to what extent the Clausius-Duhem inequality, as it is currently used, is a consequence of them.

The very fact that this question may still be raised suggests that its answer requires an examination of the foundations of thermodynamics to a depth well beyond that explored by pioneers of the subject. To cite only a few examples of studies already mounted in this direction we might mention important work by GURTIN and WILLIAMS [GW], by DAY [D1] and by COLEMAN and OWEN [CO]. Because more recent efforts by SERRIN [S1–S3] and by ŠILHAVÝ [S4] bear a particularly close relation to ours, we wish to say a few words about each.

The work contained here was motivated by a conversation with JAMES SERRIN in the late spring of 1978. At that time SERRIN explained how, for each process suffered by a body, heat exchange between the body and its exterior could be codified in the form of a signed measure on a presupposed one-dimensional *hotness manifold* endowed with a total order. Moreover, he indicated how various classical statements of the Second Law could be invoked in terms of such measures and how, by also invoking a *union axiom* and existence of certain archetypal materials, one could deduce existence and uniqueness of a continuous positive-valued function on the hotness manifold that plays the role of temperature in the Clausius inequality.

Shortly thereafter it occurred to us that, in the context of SERRIN'S formulation, similar results could be obtained directly on the basis of somewhat different hypotheses by means of those versions of the Hahn-Banach Theorem which assert that certain pairs of convex sets in a topological vector space admit separation by a hyperplane. In July of 1978 we prepared for SERRIN some notes to this effect [FL]. These contained early versions of results reported here as Theorem 4.1 (existence of Clausius temperature scales) and Theorem 9.1 (uniqueness of Clausius temperature scales). Because SERRIN'S results remained unpublished, we made no attempt to publish ours.

We mention all of this because SERRIN'S work and ours have since evolved in directions sufficiently distinct that the current difference between the two may obscure their common origin in SERRIN'S ideas as they stood in early 1978. We wish to acknowledge not only this but also the interest SERRIN has shown in our work and his encouragement of it.

At the same time we should also call attention to recent publications of ŠILHAVÝ [S4]. Working independently, he too realized that separation theorems of the Hahn-Banach type might provide the basis for certain classical results of

---

* In the second chapter of *The Character of Physical Law*, Feynman [F] suggests that extrapolations of this kind are not only common in physics but also essential to its progress.

thermodynamics. Indeed his arguments for the existence of Clausius temperature scales are similar but not identical to those in our 1978 notes. Readers interested in results contained here are encouraged to study ŠILHAVÝ's work as well.

We turn now to a description of work contained in this article. As has been mentioned, our focus will be on foundations underlying the Clausius inequality. In order that we might describe at least some of our objectives it will be useful to consider two crude and imprecise statements intended to reflect ideas normally associated with that inequality. One is a statement of *existence*, while the other is a statement of *uniqueness*.

(A) *There exists a positive-valued temperature scale that gives temperature as a function of state and, for cyclic processes, satisfies the Clausius inequality*

$$\int \frac{d\!\!\!/q}{T} \leqq 0, \tag{1.1}$$

*"$d\!\!\!/q$ denoting the element of heat received from external sources and $T$ the temperature of the part of the system receiving it."**

(B) *This temperature scale is unique (up to multiplication by a positive constant).*

Statements like these, to the extent they can be made precise and proved, take on whatever truth they have only within the framework of a well defined thermodynamical theory, and it is only within such a context that their validity can either be confirmed or denied. Thermodynamical theories, however, are of many kinds: There are theories of reversible processes suffered by arbitrary materials, theories of arbitrary processes suffered by special materials, and so on. Not only may the bodies and processes admitted for consideration vary from one theory to another, so may the manner in which the notion of *state* is rendered concrete. Truths seemingly deduced *solely* on the basis of thermodynamical laws very rarely are, for special and often tacit features of a particular theory—for example, the presumption of an adequate supply of reversible processes—are almost invariably brought into play. Thus, it becomes difficult to distinguish between those truths which, in some sense, cut across the full spectrum of thermodynamical theories and those which require for their validity special conditions prevailing only within certain theories.

With this in mind we wish to examine statements like those posited from a rather broad perspective. We believe that most, if not all, special theories based upon consideration of cyclic processes share features which, when suitably

---

* The integral is a calculation effected for the entire process. The interpretation of symbols appearing in (1.1) is taken verbatim from the opening paragraph of *On the Equilibrium of Heterogeneous Substances* [G]. There GIBBS appears to have been speaking of the more general Clausius-Duhem inequality rather than the special form (1.1) it takes for cyclic processes. Although some writers interpret the symbol $T$ in (1.1) to be the temperature of the *source* of heat (typically regarded to be a system of "heat baths"), we have invoked GIBBS' words because they are closer in spirit to the interpretation of the Clausius inequality we wish to examine.

abstracted, permit analysis of those theories within a common framework. In the spirit of work by COLEMAN and OWEN, we seek to develop a "theory of theories" which, for example, makes possible isolation of *precisely* those attributes a particular theory must have so that within its framework one or another of the two statements takes on validity. We shall be especially interested in separating those aspects of the Clausius inequality that depend *only* on the force of the Second Law from those that depend upon *additional* structure that might or might not be present within a given theory. Insofar as all theories presumably have the Second Law built into them, this will enable us to identify results which transcend the details of special theories.

In Section 2 we provide some mathematical preliminaries. In particular, we state the version of the Hahn-Banach Theorem that plays a role in proof of virtually every theorem in the main body of this article. When not used explicitly it manifests itself in the guise of Lemma 6.1 or 6.2.

In Section 3 we abstract those features of theories of cyclic processes that will concern us. It seems to us that if, within a particular theory, statements (A) and (B) can be made precise and tested for validity two things are essential. First, the notion of *state* must be rendered concrete within the theory. If temperature is to be a "function of state", then the domain of "states" on which that function is to take values should be described clearly at the outset. Second, the cyclic processes admitted for consideration within the theory should be delineated to the extent that the integral in (1.1) can, at least in principle, be calculated for each.

With these ideas in mind we take a theory to be described by specification of two sets, $\Sigma$ and $\mathscr{C}$, which carry the required information. A meaningful interpretation of these must await Section 3. Here we can only attempt a vaguely suggestive discussion.

The set $\Sigma$, called the set of *state descriptions* (or, less formally, the set of *states*), serves to specify the manner in which states of material points are described within a particular theory. Roughly speaking, elements of $\Sigma$ are the "values" states might conceivably take. Thus, in a theory of a particular gas, elements of $\Sigma$ might be pairs $(p, v)$, where $p$ is the pressure at a material point and $v$ is the specific volume. In a theory of an elastic solid, elements of $\Sigma$ might be taken to be pairs $(e, F)$, where $e$ is the internal energy density and $F$ is the deformation gradient. In a theory that takes as primitive the existence of a hotness manifold, elements of $\Sigma$ might be taken to be the "hotnesses" material points can experience.

In any case, we presume that $\Sigma$ is endowed with a Hausdorff topology. Moreover, we presume throughout the main body of this article that $\Sigma$ may be taken to be compact. In effect we restrict our attention to processes in which no material point experiences a state outside some fixed compact set, perhaps very large. That we impose this restriction at the outset results from a decision to sacrifice a degree of generality in exchange for a presentation substantially less encumbered by technical considerations. To compensate for this we relax the compactness assumption in Appendix E. There we show that, in the absence of compactness, important theorems in the main body of this article require modification, and we indicate what modifications need be made.

The set $\mathscr{C}$, called the set of *cyclic heating measures*, carries information about heat exchange in those cyclic processes a particular theory admits. With each such process there is associated a signed Borel measure on $\Sigma$ that provides an account of net heat receipt by the body suffering the process according to the states experienced by its material points as they exchange heat with the exterior of the body. More precisely, if $\mathscr{q}$ is the measure associated with a process and $\Lambda \subset \Sigma$ is a (Borel) set of states, then $\mathscr{q}(\Lambda)$ is the net amount of heat received during the entire process (from the exterior of the body) by material points experiencing states in $\Lambda$. The set of measures derived in this way from all cyclic processes admitted within a particular theory is what we call $\mathscr{C}$. This rather terse interpretation of $\mathscr{C}$ is elaborated upon considerably in Section 3, where we also attribute to $\mathscr{C}$ a natural convexity property one would expect in a reasonable theory.

The pair $(\Sigma, \mathscr{C})$ taken to characterize a particular theory we call a *cyclic heating system* (Definition 3.1). A *Kelvin-Planck system* is a cyclic heating system that, in a sense made precise in Definition 3.2, respects the Kelvin-Planck statement of the Second Law. In terms that are not quite precise a Kelvin-Planck systems is a cyclic heating system for which no nonzero cyclic heating measure takes non-negative values in every Borel set. Thus, if heat is absorbed by a body suffering a cyclic process that body must emit heat as well.

In Section 4 we prove (Theorem 4.1) that *every* Kelvin-Planck system $(\Sigma, \mathscr{C})$ admits a continuous function $T: \Sigma \to \mathbb{P}$, $\mathbb{P}$ denoting the positive real numbers, such that

$$\int_{\Sigma} \frac{d\mathscr{q}}{T} \leqq 0, \qquad \forall \mathscr{q} \in \mathscr{C}. \tag{1.2}$$

Such a function we call a *Clausius temperature scale* for the Kelvin-Planck system $(\Sigma, \mathscr{C})$. *Existence* of Clausius temperature scales follows directly from the definition of a Kelvin-Planck system by means of a straightforward application of the Hahn-Banach Theorem. *No additional apparatus (Carnot cycles, reversible processes, equilibrium states, existence of special materials) need be brought into play.* Theorem 4.1 also asserts that if, for a cyclic heating system $(\Sigma, \mathscr{C})$, there exists a continuous function $T: \Sigma \to \mathbb{P}$ that satisfies (1.2), then $(\Sigma, \mathscr{C})$ must be a Kelvin-Planck system. In this sense belief in the Kelvin-Planck Second Law, to the extent that it is expressed in Definition 3.2, is *equivalent* to belief in the *existence* of a Clausius temperature scale.

*Uniqueness* of a Clausius temperature scale is a very different matter. For a given Kelvin-Planck system $(\Sigma, \mathscr{C})$, the set of continuous functions $T: \Sigma \to \mathbb{P}$ that might satisfy condition (1.2) is clearly tied to the supply $\mathscr{C}$ of cyclic heating measures and, therefore, to the supply of cyclic processes from which they derive. In rough terms, the richer the supply of cyclic processes admitted within a particular theory the more demanding condition (1.2) becomes and the smaller will be the collection of functions that might qualify as Clausius temperature scales. Thus, to ask for a Kelvin-Planck system that all its Clausius scales be constant multiples of some fixed one is to ask that the supply of cyclic heating measures for the system be suitably rich.

Motivated in part by this idea we devote a good deal of the remainder of the article to study of the relationship between the supply of cyclic heating measures for a Kelvin-Planck system and properties of the collection of Clausius temperature scales the system admits. We postpone until Section 9 direct consideration of issues connected with uniqueness of Clausius scales, for there are other important issues that are more sensibly addressed beforehand.*

Sections 5–8 amount to an investigation of the notions of *hotness* and *hotter than*. If, for a Kelvin-Planck system $(\Sigma, \mathscr{C})$, we wish to refer to all continuous functions $T: \Sigma \to \mathrm{P}$ satisfying (1.2) as *temperature* scales, then we must show that in some sense these functions convey information about the relative hotness of material points as they manifest themselves in the various states contained in $\Sigma$. This, in turn, requires that $\Sigma$ carry some "hotness structure" wherein it becomes meaningful to say that two states in $\Sigma$ are of the same hotness or that one is hotter than another. Moreover, we would like such a structure to be posited without recourse to the existence of Clausius scales so that we may subsequently examine the extent to which the relative hotness of elements of $\Sigma$ is precisely reflected in the numbers the Clausius scales assign to them.

We wish to emphasize that we do *not* take the notions of *hotness* or *hotter than* as primitive. In particular, we do not insist for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ that elements of $\Sigma$, the set of state descriptions, carry with them a *presupposed* indicator of hotness. Rather, we take the position that questions concerning the relative hotness associated with the various states should, to the extent possible, be decided within the context of a particular theory by study of the processes admitted by the theory. For a theory of cyclic processes described by a Kelvin-Planck system $(\Sigma, \mathscr{C})$ we regard any hotness structure carried by the set $\Sigma$ to be imposed by the set $\mathscr{C}$, which codifies information about heat exchange in those processes the theory admits.

By way of introduction to our discussion of hotness we provide Section 5. There we merely review a fact which is well known to mathematicians but which is either unknown or overlooked by some writers on thermodynamics: It is not *generally* true that a total order relation on a set can be precisely reflected in a real numerical scale. Thus, if for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ the set $\Sigma$ is given some hotness structure wherein means are provided for deciding when two states are of the same hotness or when one is hotter than another, there is no reason to suppose in advance that this structure can be precisely reflected in *any* real numerical scale, much less a Clausius temperature scale. This we take as something to prove.

In Section 6 we begin our formal consideration of hotness by indicating how, for a Kelvin-Planck system $(\Sigma, \mathscr{C})$, the set $\mathscr{C}$ serves to partition the set of states $\Sigma$ into equivalence classes called *hotness levels*, each consisting of states of identical hotness. The set of hotness levels is designated by the symbol $H$ and is given the quotient topology it inherits from $\Sigma$. These constructions are effected without recourse to the existence of Clausius temperature scales. A connection between Clausius scales and hotness is drawn by Theorem 6.1. In rough terms *this theorem asserts not only that two states of identical hotness are assigned the same temperature by every Clausius scale for* $(\Sigma, \mathscr{C})$ *but also the converse: if two*

* Readers wishing to proceed quickly from the existence question taken up in Section 4 to the uniqueness question addressed in Section 9 can do so after reading Section 6. In this case Remark 9.1 can be passed over without loss of continuity.

*states are not distinguished by any Clausius scale here must exist cyclic processes such as to establish the two states to be of the same hotness.*

Having indicated in Section 6 how, for a Kelvin-Planck system $(\Sigma, \mathscr{C})$, the set $\mathscr{C}$ induces in $\Sigma$ a set $H$ of hotness levels, we consider in Section 7 how one might give meaning to the idea that states in one hotness level are hotter than those in another. There are various ways in which this might be done. In fact we examine four slightly different *hotter than* relations one can give to $H$, each defined solely in terms of the supply of cyclic processes as represented by the set $\mathscr{C}$ of cyclic heating measures. In each case we examine how the resulting hotness structure on $H$ is reflected in the collection of Clausius temperature scales admitted by the Kelvin-Planck system $(\Sigma, \mathscr{C})$. In very rough terms, the theorems of Section 7 not only indicate what properties each hotness structure imposes upon the Clausius scales, they also assert that *if the Clausius scales assign temperatures to hotness levels in a certain way then the supply of cyclic processes must be sufficiently rich as to impose a corresponding hotness structure on H.*

In order to understand the purpose of Section 8 it is important to keep in mind two things. The first is that, as an example in Section 5 demonstrates, a set endowed with a total order need not be order-similar to a subset of the real line. The second is that, in the context of this study, the set $H$ of hotness levels for a Kelvin-Planck system emerges from Section 6 as a defined object: the hotness levels are equivalence classes of states in $\Sigma$ induced solely by the supply $\mathscr{C}$ of cyclic heating measures, and the set $H$ inherits its topology from that of $\Sigma$ in the usual way. Because nothing is presumed about the topology of $\Sigma$ other than it be compact and Hausdorff, there is not much one can say of a general nature about the topology of $H$ other than that it too must be compact and Hausdorff.

In Section 8, however, we show the following: If $\Sigma$ is a metric space and $\mathscr{C}$ is sufficiently rich in cyclic heating measures as to totally order $H$ even with respect to the weakest *hotter than* relation defined in Section 7 *then it can only be the case that H is both homeomorphic and order-similar to a subset of the real line*; *in particular, if $\Sigma$ is connected then the set of hotness levels must be homeomorphic and order-similar to an interval of the real line.* These same results obtain whether or not $\Sigma$ is a metric space provided that $H$ is totally ordered by any but the weakest of the *hotter than* relations discussed in Section 7; moreover, *every Clausius scale reflects the order precisely.*

Finally we turn in Section 9 to consideration of those properties a Kelvin-Planck system must possess in order that all its Clausius temperature scales be identical (up to multiplication by a positive constant). We remarked earlier that in order for all the Clausius scales for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ to be essentially identical the set $\mathscr{C}$ of cyclic heating measures should, in some sense, be suitably large. This is to say that any theory purporting to yield an essentially unique Clausius temperature scale should admit an appropriately large supply of cyclic processes. Indeed the classical argument for uniqueness (and existence) of a Clausius temperature scale presumes that, for any pair of hotness levels one might wish to consider, a Carnot cycle can be made to operate between them. Thus, the classical argument suggests that a suitably rich supply of (reversible) Carnot cycles is *sufficient* to ensure that all Clausius temperature scales are constant multiples of some fixed one.

In Theorem 9.1 we assert that for all Clausius scales for a Kelvin-Planck system to be essentially identical it is not only sufficient *but also necessary* that the system be equipped with Carnot elements operating between every distinct pair of hotness levels. (In terms that are not quite accurate, a Carnot element for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ is a cyclic heating measure in $\mathscr{C}$ whose negative is also in $\mathscr{C}$ and which characterizes a process wherein heat is absorbed entirely within one hotness level and emitted entirely from another.) In a sense made precise by Remark 9.2 we also assert that for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ to admit an essentially unique Clausius scale it is not only necessary that every hotness level manifest itself in some Carnot element *so must every state in* $\Sigma$. The corollaries to Theorem 9.1 draw additional conclusions about the supply of reversible and irreversible processes in Kelvin-Planck systems for which all Clausius scales are positive multiples of some fixed one.

Existence and uniqueness of Clausius scales are, of course, very different things, and it should come as no surprise that conditions necessary for one are largely superfluous to the other. Theorem 4.1. ensures the *existence* of at least one Clausius temperature scale for *any* Kelvin-Planck system. In this sense *existence* of Clausius temperature scales follows directly from the Second Law (as expressed by Definition 3.2) *without the intervention of Carnot cycles, reversible processes, quasi-static processes or any other conceptual apparatus normally built into standard existence arguments.* There is nothing in Theorem 4.1 to support the position of those orthodox thermodynamicists who would argue that a Clausius temperature scale can have as its domain only those states which might manifest themselves during the course of a reversible process. These same thermodynamicists should, however, find comfort in Theorem 9.1, which suggests that the presence of reversible processes—and Carnot cycles in particular—is inextricable in any theory that yields for a Kelvin-Planck system an essentially *unique* Clausius temperature scale. Moreover, Remark 9.2 suggests that in such a theory every state in the domain of the Clausius scale *must* manifest itself in some Carnot cycle. Thus, it appears that orthodox critticism should be directed not at those applications which require only the *existence* of a Clausius temperature scale but rather at those which invoke *uniqueness* as well.*

In Section 10 we make some concluding remarks. In particular we briefly discuss prospects for future work on foundations underlying the Clausius-Duhem inequality.

We conclude this section with some words of advice for physical scientists who might wish to study ideas contained here but who, in glancing at the pages of this article, feel insufficiently trained in modern mathematics to attempt even

* The assertions made in this paragraph, like all others in this introduction, are intended only to convey the general sense of our results; they are not intended to be precise and complete statements of them. In particular, we wish to remind readers that compactness of the set of states is presumed for every Kelvin-Planck system considered in the main body of this article. When the compactness assumption is relaxed, questions concerning both existence and uniqueness of Clausius scales become somewhat more complicated; these are taken up in Appendix E.

a superficial reading. In fact, the amount of mathematical knowledge needed to achieve a reasonable understanding of this work is not so formidable as might be supposed.

Readers equipped with some knowledge of modern linear algebra should be able to grasp the outline of the theory once it is understood that the space $\mathcal{M}(\Sigma)$ in which we work is a vector space that has many features in common with the more familiar vector space $\mathbb{R}^n$. Indeed, the Hahn-Banach Theorem serves to specify some of those common features. At least with respect to certain issues, then, arguments about $\mathcal{M}(\Sigma)$ can be made in the same way they are made about $\mathbb{R}^n$.

While vectors in $\mathbb{R}^n$ are sequences of real numbers, vectors in $\mathcal{M}(\Sigma)$ are real-valued Borel measures on a topological space $\Sigma$. Thus, the reader should know something about topological spaces and what one means by a real Borel measure on such a space. The reader should also know something about integration of continuous functions with respect to Borel measures. *However, a rudimentary knowledge of these things is all that is required for a reasonable understanding of this article.*

Pages 5–21 and 34–39 of the book by RUDIN [R1] provide a brief introduction to topology, measure, and integration. The emphasis there is on positive measures. We shall also deal with real measures (that is, measures that take both positive and negative values), but the reader can think of a real measure as one obtained by taking the difference of two positive measures. Once it is clear what is meant by a real measure on a topological space $\Sigma$, he should have little difficulty seeing that the set $\mathcal{M}(\Sigma)$ of all such measures, equipped with the obvious rules for addition and for multiplication by a real number, is in fact a vector space. Useful facts about $\mathcal{M}(\Sigma)$ are provided in the next section.

## 2. Notation and Mathematical Preliminaries

We denote the real numbers by $\mathbb{R}$ and the positive real numbers by $\mathbb{P}$.

A relation $\succ$ on a set $X$ is a **partial order** on $X$ if $\succ$ is both

$$\textit{transitive}: \; x \succ y, y \succ z \Rightarrow x \succ z$$

and

$$\textit{antisymmetric}: \; x \succ y \Rightarrow y \not\succ x.$$

If $\succ$ is a partial order on a set $X$ there may exist elements $x$ and $y$ in $X$ for which we have neither $x \succ y$ nor $y \succ x$. If we have either $x \succ y$ or $y \succ x$ then we say that $x$ and $y$ are **$\succ$-comparable.** If all distinct $x$ and $y$ are $\succ$-comparable then $\succ$ is a **total order** on $X$. When $X$ and $X^*$ are sets endowed with partial orders $\succ$ and $^*\!\succ$, respectively, then $X$ and $X^*$ are **order-similar** if there exists a bijective map $f: X \to X^*$ such that

$$x \succ y \Leftrightarrow f(x) \,^*\!\!\succ f(y).$$

Let $V$ be a vector space.† A set $U \subset V$ is a **cone** if

$$x \in U, \alpha \in \mathbb{P} \Rightarrow \alpha x \in U.$$

The **cone generated by a set** $U \subset V$, denoted Cone $(U)$, is defined by

$$\text{Cone}\,(U) := \{\alpha x \in V \mid x \in U, \alpha \in \mathbb{P}\}.$$

A set $U \subset V$ is **convex** if

$$x \in U, y \in U, \lambda \in [0, 1] \Rightarrow \lambda x + (1 - \lambda) y \in U.$$

By the **convex hull** of a set $U \subset V$ we mean the smallest convex set in $V$ that contains $U$. If $U \subset V$ is a cone, then $U$ is convex if and only if

$$x \in U, y \in U \Rightarrow x + y \in U.$$

A set $U \subset V$ is a **linear subspace** if

$$x \in U, y \in U, \alpha \in \mathbb{R}, \beta \in \mathbb{R} \Rightarrow \alpha x + \beta y \in U.$$

Every linear subspace is a convex cone. The **span** of a set $U \subset V$ is the smallest linear subspace that contains $U$. A set $U \subset V$ is an **affine hyperplane** if there exists a non-zero linear function $f: V \to \mathbb{R}$ and a number $\gamma \in \mathbb{R}$ such that

$$U = \{x \in V \mid f(x) = \gamma\},$$

and we say that the sets

$$\{x \in V \mid f(x) \geqq \gamma\} \text{ and } \{x \in V \mid f(x) \leqq \gamma\}$$

are the (opposite) **half spaces** of the affine hyperplane $U$. An affine hyperplane $U$ **separates** two sets $A \subset V$ and $B \subset V$ if $A$ lies in one half space of $U$, $B$ lies in the other, and either $A$ or $B$ fails to meet $U$; the separation is **strict** if neither $A$ nor $B$ meets $U$.

A **topological vector space** is a vector space $V$ endowed with a topology such that the maps

$$(x, y) \in V \times V \mapsto x + y \in V$$

and

$$(\alpha, x) \in \mathbb{R} \times V \mapsto \alpha x \in V$$

are continuous. If $V$ is a topological vector space we denote by $c\ell(U)$ the topological closure of the set $U \subset V$. When $U$ is a cone, $c\ell(U)$ is also a cone; when $U$ is convex, $c\ell(U)$ is also convex; and when $U$ is a linear subspace $c\ell(U)$ is also a linear subspace. We use the symbol $\hat{U}$ to denote the closure of the cone generated by $U$; that is,

$$\hat{U} := c\ell(\text{Cone}(U)).$$

Since the closure of a cone is also a cone it follows that, for each set $U$ in a topological vector space, $\hat{U}$ is a closed cone. A Hausdorff topological vector space is one in which the topology is Hausdorff. In such spaces the convex hull of the

† All vector spaces considered here are real vector spaces.

union of a pair of compact convex sets is again compact. A topological vector space is **locally convex** if every neighborhood of the zero vector contains a convex neighborhood of the zero vector.

There are several versions of the Hahn-Banach Theorem. Those we shall use are of the kind which assert that certain disjoint pairs of convex sets can be separated by a closed affine hyperplane. Although another version will find use in our appendices, we state here the only version we shall use in the main body of this article.

**Theorem 2.1.** (*Hahn-Banach*) *Let $V$ be a Hausdorff locally convex topological vector space, and let $A$ and $B$ be non-empty disjoint closed convex subsets of $V$ with $B$ compact. Then there exists a continuous linear function $f: V \to \mathbb{R}$ and a number $\gamma \in \mathbb{R}$ such that*

$$f(a) < \gamma \text{ for all } a \in A$$

*and*

$$f(b) > \gamma \text{ for all } b \in B.$$

*In particular, if $A$ is a cone then*

$$f(a) \leqq 0 \text{ for all } a \in A$$

*and*

$$f(b) > 0 \text{ for all } b \in B.$$

The last sentence of Theorem 2.1 is usually not made explicit, but it is an easy consequence of the sentence preceding it. In fact we shall be concerned almost exclusively with the situation in which $A$ is a cone. If $A$ is not only a cone but also a linear subspace then the fact that $-a$ is a member of $A$ for every $a \in A$ implies that the function $f$ in Theorem 2.1 must take the value zero everywhere on $A$.

Let $\Sigma$ be a *compact* Hausdorff space. By $C(\Sigma, \mathbb{R})$ we mean the vector space of continuous real-valued functions on $\Sigma$, and we denote by $C(\Sigma, \mathbb{P})$ the subset of $C(\Sigma, \mathbb{R})$ consisting of those functions that take strictly positive values. By $\mathscr{M}(\Sigma)$ we mean the vector space of real (signed) regular Borel measures on $\Sigma$, and by $\mathscr{M}_+(\Sigma)$ we mean the convex cone in $\mathscr{M}(\Sigma)$ consisting of those measures that take non-negative values on every Borel set in $\Sigma$. Note that $\mathscr{M}_+(\Sigma)$ contains the zero measure. By $\mathscr{M}^1_+(\Sigma)$ we mean the set of measures in $\mathscr{M}_+(\Sigma)$ of mass one; that is,

$$\mathscr{M}^1_+(\Sigma) = \{\nu \in \mathscr{M}_+(\Sigma) \mid \nu(\Sigma) = 1\}.$$

It is easy to confirm that $\mathscr{M}^1_+(\Sigma)$ is convex. For each $\sigma \in \Sigma$ we denote by $\delta_\sigma$ the **Dirac measure** concentrated at $\sigma$; that is, for each Borel set $B \subset \Sigma$

$$\delta_\sigma(B) = \begin{Bmatrix} 1 & \text{if} & \sigma \in B \\ 0 & \text{if} & \sigma \notin B \end{Bmatrix}.$$

Clearly, for each $\sigma \in \Sigma$ we have that $\delta_\sigma$ is a member of $\mathscr{M}^1_+(\Sigma)$. By the **support** of a measure $\nu \in \mathscr{M}_+(\Sigma)$, denoted $\operatorname{supp} \nu$, we mean the complement in $\Sigma$ of the largest open set of $\nu$-measure zero. In particular, $\operatorname{supp} \delta_\sigma = \{\sigma\}$.

*Throughout this article it will be understood that* $\mathscr{M}(\Sigma)$ *is given the* **weak-star topology** (*sometimes called the* **vague topology**). This is defined as follows: For each $\phi \in C(\Sigma, \mathbb{R})$ let $\tilde{\phi}: \mathscr{M}(\Sigma) \to \mathbb{R}$ be the linear transformation defined by

$$\tilde{\phi}(\nu) \equiv \int_{\Sigma} \phi \, d\nu .$$

The weak-star topology on $\mathscr{M}(\Sigma)$ is the coarsest topology that renders $\tilde{\phi}$ continuous for every $\phi \in C(\Sigma, \mathbb{R})$. Endowed with the weak-star topology, $\mathscr{M}(\Sigma)$ has certain properties we shall find useful: First, $\mathscr{M}(\Sigma)$ is a Hausdorff locally convex topological vector space. Second, every continuous real-valued linear function on $\mathscr{M}(\Sigma)$ is of the kind $\tilde{\phi}$ for some $\phi \in C(\Sigma, \mathbb{R})$; that is, if $f: \mathscr{M}(\Sigma) \to \mathbb{R}$ is continuous and linear then there exists a (unique) $\phi \in C(\Sigma, \mathbb{R})$ such that

$$f(\nu) \equiv \int_{\Sigma} \phi \, d\nu .$$

Finally, $\mathscr{M}^1_+(\Sigma)$ is compact.

A useful reference for all the material contained in this section is the set of lectures by CHOQUET [C1].

## 3. Cyclic Heating Systems and Kelvin-Planck Systems

Our entire study amounts to an investigation of a class of mathematical objects called *Kelvin-Planck systems*. These in turn are members of a wider class of objects called *cyclic heating systems*. In essence, a Kelvin-Planck system is a cyclic heating system that is compatible with the Kelvin-Planck statement of the Second Law.

We begin with formal statements of what we mean by a cyclic heating system and by a Kelvin-Planck system. Although both definitions are brief they will appear somewhat opaque at first glance. The balance of this section, however, will be devoted to informal discussion in which we hope to make more transparent the physical ideas our definitions are intended to carry.

It is worth repeating here that if $\Sigma$ is a compact Hausdorff space then $\mathscr{M}(\Sigma)$ is the vector space of (signed) Borel measures on $\Sigma$ and $\mathscr{M}_+(\Sigma)$ is the subset of $\mathscr{M}(\Sigma)$ consisting of all Borel measures on $\Sigma$ that take non-negative values on each Borel set.

**Definition 3.1.** *A* **cyclic heating system** *consists of two non-empty sets*:

(i) *a set* $\Sigma$ *endowed with a compact Hausdorff topology. Elements of* $\Sigma$ *are called* **state descriptions** *or, less formally,* **states**.

(ii) *a set* $\mathscr{C} \subset \mathscr{M}(\Sigma)$ *such that the cone* $\hat{\mathscr{C}} \subset \mathscr{M}(\Sigma)$ *defined by*

$$\hat{\mathscr{C}} = \mathscr{cl}\,(\mathrm{Cone}(\mathscr{C}))$$

*is convex. Elements of* $\mathscr{C}$ *are called* **cyclic heating measures.**

**Definition 3.2.** *A* **Kelvin-Planck system** *is a cyclic heating system* $(\Sigma, \mathscr{C})$ *such that*

$$\hat{\mathscr{C}} \cap \mathscr{M}_+(\Sigma) = \{0\}.$$

We begin our discussion of these definitions with a caveat. Words like *body*, *material point*, and *process* do not appear in Definitions 3.1 and 3.2, nor will they appear in any subsequent definitions, theorems, or proofs. They will, however, be used in this section and indeed throughout our entire narrative, but never in an official capacity. Rather, they will be invoked in an informal way only to guide the reader in connecting definitions with physical experience and to lend interpretation to results that flow from those definitions. In any case these words will always stand in relation to each other in a prescribed way: Bodies are deemed to be composed of material points, and processes are suffered by bodies.

A thermodynamical theory is, for the most part, a theory of a particular universe of material bodies taken together with specification of a class of processes those bodies might admit. Such a theory might be of the grandest scope, as when all possible processes suffered by all possible bodies are admitted for consideration. On the other hand, such a theory might be a more restricted one in which the universe of bodies considered is circumscribed, in which the processes studied are circumscribed, or in which circumscriptions of both kinds are in force.

Because our interest here is in the Clausius inequality our focus will be on theories concerned with cyclic processes—that is, with processes in which the body experiencing the process is, in some sense, in the same condition at both the beginning of the process and at its end. The universe of bodies under study might be entirely general or it might be restricted to those bodies composed of a particular material. The latter situation is the one we shall have most prominently in mind as we lend interpretation to definitions and theorems.

We believe that most, if not all, theories based upon consideration of cyclic processes share underpinnings which, if properly abstracted, permit analysis of such theories within a common framework. Our definition of a *cyclic heating system* is intended to abstract those objects which, for the purposes we have in mind, serve to specify a particular theory.

The first of these objects is a set denoted by $\Sigma$ and called *the set of state descriptions*. The bodies under consideration in a particular theory are composed of material points, and we take as a primitive idea that at each instant every material point manifests itself in some state. Note that we shall not confine our attention to theories of homogeneous bodies. That is, we admit the possibility that at some instant two material points within a given body might be in distinct states. In any case, *we regard as an intrinsic part of a theory the specification of the manner in which states of material points are afforded mathematical description.* By $\Sigma$ we mean the set of "values" these states* might possibly take in a particular theory under consideration. Some examples will help clarify how $\Sigma$ is to be interpreted.

Consider a theory concerned with some universe of bodies, and imagine that the theory presupposes the existence of a real-valued empirical temperature scale. Then, for the purposes of the theory, the state of a material point might be deemed to be suitably described solely by its temperature on that scale. Suppose further that the processes under consideration are all cyclic processes wherein

* The word *state* will always refer to the condition of a material point within a body. It will never be used in reference to the condition of the body as a whole.

no material point experiences an empirical temperature below some value $\alpha$ or above some value $\beta$. In this case $\Sigma$ would be identified with the closed interval $[\alpha, \beta]$, which could be very large.

In a theory of processes suffered by homogeneous bodies composed of a particular gas the instantaneous state of a material point might be deemed to be described adequately by specification of instantaneous values at that point of both the pressure, $p$, and the specific volume, $v$. Thus, a state description would be an element $(p, v) \in \mathbb{P} \times \mathbb{P}$. The processes under consideration might be those in which no material point experiences a pressure or a specific volume such that $(p, v)$ lies outside some closed and bounded region $K \subset \mathbb{P} \times \mathbb{P}$, perhaps very large. In this case $\Sigma$ would be identified with the set $K \subset \mathbb{P} \times \mathbb{P}$.

In a theory of bodies composed of a particular elastic material the instantaneous state of a material point might be deemed to be described adequately by specifying for that point instantaneous values of both the internal energy per unit mass, $e$, and the deformation gradient, $F$, relative to some reference configuration. Here $F$ is a member of Lin $(\mathscr{V})$, the vector space of linear transformations on three-dimensional Euclidean space, $\mathscr{V}$. In this case the state of a material point would be identified with an element $(e, F)$ in the vector space $\mathbb{R} \oplus \text{Lin}(\mathscr{V})$, which we presume to be endowed with a suitable norm. If the processes under consideration are such that no material point experiences a state outside some closed and bounded region $D \subset \mathbb{R} \oplus \text{Lin}(\mathscr{V})$, perhaps very large, we would identify $\Sigma$ with $D$.

In any case we shall assume that $\Sigma$, the set of state descriptions, is equipped with a Hausdorff topology. Moreover, as in the preceding examples *we shall assume that $\Sigma$ may be taken to be compact.* In effect, then, we shall restrict our attention to processes in which no material point experiences a state outside some fixed compact set, perhaps very large. This presumption is built into Definition 3.1 in order that it might be stated explicitly at the outset once and for all.

We do not think it unreasonable to restrict attention to processes in which the states material points experience are in some sense circumscribed. For example, it is common for a thermodynamic theory of a particular material to be of the kind in which states of material points are identified with elements of some finite-dimensional Banach space. Moreover, the material under study is usually presumed to be characterized by a collection of constitutive "functions of state" that fix certain attributes of a material point once its state has been specified. Implicit in this presumption is the idea that such functions have associated with them a *constitutive domain*—that is, a subset of the ambient Banach space corresponding to states in which the posited constitutive functions (and indeed the very means whereby states are described) are appropriate to the intended range of the theory.

It is clear, then, that processes admitted for study should be limited to those in which material points experience no state outside some region in the constitutive domain. To require this region to be compact need not be unduly restrictive: If the constitutive domain is bounded and is not terribly pathological, then the compact region in question might be thought of as one that closely approximates the entire constitutive domain.

That we require $\Sigma$ to be compact results from our decision to present a clean and relatively simple study of reasonable breadth as against a more technical one of even wider range. *We must however caution the reader that, without modification, important theorems stated in the main body of this article become false if the presumption that $\Sigma$ be compact is omitted.* With this in mind we provide Appendix E, in which we take up issues connected with the weaker requirement that $\Sigma$ be *locally* compact.

**Remark 3.1.** We should acknowledge the importance of theories in which states of material points are identified with elements of an infinite-dimensional Banach space. We have in mind thermodynamic theories of materials with memory in which states are identified with a suitably normed space of history functions [C2]. At least with respect to the norm topology such spaces are not locally compact, and compact sets within them have empty interior. In this case, it is somewhat more difficult to justify a study of cyclic processes in which material points experience no state outside some fixed compact set. For materials with memory, however, our compactness assumption may be less an issue than is the existence of a supply of precisely cyclic processes rich enough to support a meaningful theory based solely on study of them.

Having discussed the first set, $\Sigma$, required for specification of a cyclic heating system, we turn now to consideration of the second set mentioned in Definition 3.1—the set $\mathscr{C}$ of *cyclic heating measures.* While $\Sigma$ indicates the means by which states of material points are afforded description in a particular theory, the set $\mathscr{C} \subset \mathscr{M}(\Sigma)$ serves to describe features of the cyclic processes the theory admits. To each such process there corresponds a cyclic heating measure which, for the purposes we have in mind, encodes the nature of heat exchange between the body experiencing the process and its exterior.

Because of its importance to all that we do, our discussion of cyclic heating measures will be extensive. Here we indicate briefly what outline that discussion will follow. First we indicate what we mean by a *heating measure* for a process (not necessarily cyclic), and immediately thereafter we provide two broad-based examples to illustrate how heating measures are constructed for processes. Then we indicate what structure we expect of the collection of heating measures corresponding to all cyclic processes deemed to be admitted within a particular theory.

Consider a theory concerned with a particular universe of bodies, and suppose that states of material points are identified with elements of a Hausdorff topological space. The bodies in question admit certain processes; and, as before, we confine our attention to those processes in which no material point experiences a state outside some fixed compact set $\Sigma$. Now we focus on one such process experienced by some body in the universe under consideration. During the course of the process material points may suffer changes of state and, moreover, the body may receive heat from its exterior and emit heat to it. We would like to have available a mathematical device for describing in a suitably refined way the nature of heat exchange between the body and its exterior. In particular, we would like that device to provide an account of net heat receipt *according to the states experienced by material points as they exchange heat with the exterior of the body.*

For this purpose we associate with the process a *heating measure* $\mathscr{q} \in \mathscr{M}(\Sigma)$ with the following interpretation: If $\Lambda \subset \Sigma$ *is a Borel set of states then* $\mathscr{q}(\Lambda)$ *is the net amount of heat received (from the exterior of the body) during the entire process by material points experiencing states contained within* $\Lambda$. In particular, $\mathscr{q}(\Sigma)$ is the net amount of heat absorbed by the body during the course of the process.

In order that the notion of heating measure might be made more concrete we indicate how heating measures are constructed for processes in two somewhat different contexts.

**Example 3.1.** (*Nineteenth Century Thermodynamics of Homogeneous Processes*). Consider a theory concerned only with processes suffered by *homogeneous* bodies composed of a certain gas. That is, the processes under study are such that at each instant all material points of the body suffering the process are in the same state. We suppose that states of material points are identified with elements $(p, v) \in \mathrm{P} \times \mathrm{P}$, where $p$ denotes the pressure and $v$ the specific volume.* Moreover, we presume that processes admitted for consideration are restricted to those in which no material point experiences a state outside some fixed compact connected set $K \subset \mathrm{P} \times \mathrm{P}$. Thus, we identify $\Sigma$ with $K$.

We take each process to be characterized by a smooth function $\sigma : \mathscr{I} \to \Sigma$, where $\mathscr{I}$ is a closed interval of the real line. Here $\mathscr{I}$ may be construed to be the time interval during which the process takes place, and the function $\sigma(\cdot)$ may be interpreted as one which gives the state $\sigma(t) = (p(t), v(t))$ at each instant $t \in \mathscr{I}$.**

For the gas we presume the existence of constitutive functions $f : \Sigma \to \mathbb{R}$ and $g : \Sigma \to \mathbb{R}$, both continuous, such that for any process, characterized say by $\sigma(\cdot) = (p(\cdot), v(\cdot))$, the net rate of heat receipt by the body suffering the process is at each instant

$$M[f(p(t), v(t))\, \dot{p}(t) + g(p(t), v(t))\, \dot{v}(t)].$$

Here $M$ denotes the mass of the body and the dot indicates differentiation.***

---

* The theories we have in mind here are those considered by the 19th century pioneers of thermodynamics as represented in the monograph by Truesdell and Bharatha [TB]. There states are identified with pairs $(\Theta, v)$ where $\Theta$ denotes a temperature and $v$ the specific volume. In our example we choose to identify states with elements $(p, v)$ only to suggest that state descriptions need not carry with them a preconceived notion of hotness. For us hotness will emerge as a defined entity.

** This interpretation is given for didactic purposes only. In fact, $\sigma : \mathscr{I} \to \Sigma$ need only be regarded as a parameterization of the oriented curve $\sigma(\mathscr{I})$. That we require $\sigma(\cdot)$ to be a smooth function does not imply that the curve $\sigma(\mathscr{I})$ need be smooth.

*** Consider for example a perfect gas such that the isochoric molar specific heat, $c_v$, is independent of state. Then

$$f(p, v) = \frac{c_v v}{R} \quad \text{and} \quad g(p, v) = \left(\frac{c_v}{R} + 1\right) p,$$

where $R$ is the perfect gas constant.

Now consider a process suffered by a body of mass $M$, and suppose the process is characterized by the function $\sigma: \mathscr{I} \to \Sigma$. Thus if $\Lambda \subset \Sigma$ is a Borel set of states, $\sigma^{-1}(\Lambda)$ is the (Lebesgue measurable) set of instants at which material points experience states within $\Lambda$. We construct the *heating measure*, $q \in \mathscr{M}(\Sigma)$, for the process as follows: For each Borel set $\Lambda \subset \Sigma$ let

$$q(\Lambda) = M \int_{\sigma^{-1}(\Lambda)} [f(p(t), v(t))\dot{p}(t) + g(p(t), v(t))\,\dot{v}(t)]\,dt.$$

Thus, $q(\Lambda)$ is the net amount of heat received by the body only at those instants during which its material points experience states contained in $\Lambda$.

In this example the state of a material point is specified by two scalar attributes, the pressure and the specific volume. For theories of homogeneous processes in which the state or a material point is specified by several scalar attributes construction of heating measures for processes proceeds in the same way.

**Example 3.2.** (*Continuum Thermodynamics*). Consider a theory concerned with a universe of bodies, say all bodies composed of a prescribed material, and suppose that states of material points are identified with elements of a Hausdorff space. The processes under study are restricted to those in which no material point experiences a state outside some fixed compact set $\Sigma$.

Here we take a body to be a set $\mathscr{B}$ (of material points), and *parts* of the body are identified with members of a $\sigma$-algebra of subsets of $\mathscr{B}$. Now we consider a process suffered by body $\mathscr{B}$. With this process we associate:

(i) a closed interval $\mathscr{I} \subset \mathbb{R}$, to be interpreted as the time interval during which the process takes place;

(ii) a real-valued (signed) measure $h$ on $\mathscr{B} \times \mathscr{I}$ with the following interpretation: for each part $P \subset \mathscr{B}$ and each Lebesgue measurable set $I \subset \mathscr{I}$, $h(P \times I)$ is the net heat receipt (from the exterior of $\mathscr{B}$) by part $P$ during instants contained in $I$;*

(iii) a measurable function $\sigma: \mathscr{B} \times \mathscr{I} \to \Sigma$, where $\sigma(X, t)$ is to be interpreted as the state of material point $X$ at instant $t$.

The *heating measure*, $q \in \mathscr{M}(\Sigma)$, for the process is constructed as follows: For each Borel set $\Lambda \subset \Sigma$

$$q(\Lambda) = h(\sigma^{-1}(\Lambda)).$$

In rough terms, then, $q(\Lambda)$ is the net amount of heat received (during the process from the exterior of $\mathscr{B}$) only by parts of $\mathscr{B}$ experiencing states contained within $\Lambda$.

Next we consider what structure might be expected of the *collection* of heating measures corresponding to *cyclic* processes within a particular theo-

* The means we employ here for describing heat exchange between parts of the body and its exterior is similar to that employed on various occasions by GURTIN, NOLL, and WILLIAMS [GW, N1].

ry.* Thus, we consider a theory concerned with a universe of bodies, and we suppose that states of material points are identified with elements of a Hausdorff space. The cyclic processes under consideration are, as before, restricted to those in which no material point experiences a state outside a fixed compact set $\Sigma$.

For the purposes of the discussion immediately following we identify such a process with a pair $(\tau, \mathscr{q}) \in \mathrm{P} \times \mathscr{M}(\Sigma)$. Here $\tau$ is to be interpreted as the *duration* of the process—that is, the time elapsed between the beginning of the process and its end, and $\mathscr{q}$ is to be interpreted as the *heating measure* for the process. Thus, for our immediate purposes the collection of cyclic processes will be identified with a set $\mathscr{P}_c \subset \mathrm{P} \times \mathscr{M}(\Sigma)$.

We presume the set $\mathscr{P}_c$ to have certain properties:

*Property 1.* If $(\tau, \mathscr{q})$ and $(\tau, \mathscr{q}')$ are members of $\mathscr{P}_c$, then $(\tau, \mathscr{q} + \mathscr{q}')$ is also a member of $\mathscr{P}_c$.

Property 1 is a feature one would expect, for example, in a theory compatible with what SERRIN has called the *union axiom* [S1–3]. This might be explained informally in the following way: Suppose that $(\tau, \mathscr{q}) \in \mathscr{P}_c$ corresponds to a cyclic process suffered by one body in the theory under consideration and that $(\tau, \mathscr{q}') \in \mathscr{P}$ corresponds to a cyclic process suffered by another such body, not necessarily distinct from the first. Because the two processes have the same duration we can execute the two processes simultaneously (using perhaps copies of the original bodies), one in Paris and the other in Rome. In this case the "union" of the two separate bodies may be viewed as a third body which has suffered a cyclic process of duration $\tau$ with heating measure $\mathscr{q} + \mathscr{q}'$. Thus, we would expect $(\tau, \mathscr{q} + \mathscr{q}')$ to be a member of $\mathscr{P}_c$.**

Of course in this discussion we are presuming that the "union" of any two bodies, both admitted for consideration in the theory under study, is again a body admitted for consideration. Moreover, we are presuming that whenever two processes (of the same duration) are admitted for consideration in the theory, then so is the "union process" constructed from them as indicated. It is this pair of presumptions that lies at the heart of SERRIN's *union axiom*.

*Property 2.* If $(\tau, \mathscr{q})$ is a member of $\mathscr{P}_c$ and $n$ is a positive integer, then $(n\tau, n\mathscr{q})$ is also a member of $\mathscr{P}_c$.

---

* The precise sense in which the term "cyclic process" is to be understood will be less important than will be the properties of the collection of heating measures we deem to correspond to cyclic processes. Readers who wish to do so might suppose that by a cyclic process we mean one in which each material point is in the same condition at the end of the process as at its beginning. This is easy to make precise in the contexts of Examples 3.1 and 3.2.

** We required the simultaneous processes to be run in Paris and Rome only to suggest that the processes be executed in such a manner that the two bodies exchange no heat with each other.

Here we invoke the following idea: If $(\tau, \mathscr{q}) \in \mathscr{P}_c$ corresponds to a cyclic process admitted in the theory under consideration, then a new process constructed from the first by repeating it $n$ times in succession should also be admitted. The resulting process would be of duration $n\tau$ and have associated with it the heating measure $n\mathscr{q}$. Thus we should expect $(n\tau, n\mathscr{q})$ to be a member of $\mathscr{P}_c$.

*Property 3.* Let $(\tau, \mathscr{q})$ be a member of $\mathscr{P}_c$, and let $\Omega \subset \mathscr{M}(\Sigma)$ be a neighborhood of $\mathscr{q}$. Then there exists a neighborhood of $\tau$, say $\omega \subset \mathrm{P}$, with the following property: For each $\tau' \in \omega$, $(\tau', \mathscr{q}')$ is a member of $\mathscr{P}_c$ for some $\mathscr{q}' \in \Omega$.

Property 3 has the following rough interpretation: If, in the theory under consideration, there exists a cyclic process of duration $\tau$ with heating measure $\mathscr{q}$ and if $\tau'$ is a number close to $\tau$ the theory should also admit a cyclic process of duration $\tau'$ with heating measure $\mathscr{q}'$ close to $\mathscr{q}$.

Now we denote by $\mathscr{C}$ the collection of *cyclic heating measures* for the theory under consideration. That is,

$$\mathscr{C} = \{\mathscr{q} \in \mathscr{M}(\Sigma) \mid (\tau, \mathscr{q}) \in \mathscr{P}_c \text{ for some } \tau \in \mathrm{P}\}.$$

Thus, $\mathscr{C}$ is the set of all elements in $\mathscr{M}(\Sigma)$ that are heating measures for cyclic processes. If $\mathscr{P}_c$ enjoys Properties 1–3, then these impose a certain structure on $\mathscr{C}$. This is the subject of the following proposition, proof of which is provided in Appendix A.

**Proposition 3.1.** *Let $\Sigma$ be a compact Hausdorff space, let $\mathscr{P}_c \subset \mathrm{P} \times \mathscr{M}(\Sigma)$ be a set having Properties* 1–3, *and let $\mathscr{C} \subset \mathscr{M}(\Sigma)$ be defined as above. Then the cone $\hat{\mathscr{C}} \subset \mathscr{M}(\Sigma)$ defined by*

$$\hat{\mathscr{C}} = cl\,[\text{Cone}\,(\mathscr{C})]$$

*is convex.*

This completes our discussion of Definition 3.1. In summary, then, specification of a *cyclic heating system* amounts to specification of those objects in a particular theory of cyclic processes that will be of interest to us. The (compact) set $\Sigma$ indicates the means whereby states of material points are afforded mathematical description, and the set $\mathscr{C} \subset \mathscr{M}(\Sigma)$ indicates the heating measures that correspond to cyclic processes the theory admits. That we require $\hat{\mathscr{C}}$ to be convex follows from Proposition 3.1 and the discussion preceding it.

Before turning to Definition 3.2 we wish to point out that, in the context of the discussion preceding Proposition 3.1, some readers might have found it natural to require still another property of the set $\mathscr{P}_c$:

*Property 4.* If $(\tau, \mathscr{q})$ is a member of $\mathscr{P}_c$ and $\alpha$ is any positive number, then $(\tau', \alpha\mathscr{q})$ is also a member of $\mathscr{P}_c$ for some $\tau' \in \mathrm{P}$.

Were $\alpha$ restricted to be a positive integer Property 4 would amount to little more than an easy consequence of Property 1. The requirement that Property 4 hold for any $\alpha \in \mathrm{P}$ might be supported with an argument of the following kind:

If $(\tau, q)$ corresponds to a cyclic process experienced by some body, then for any $\alpha \in \mathbb{P}$ there should exist a "similar" cyclic process, experienced perhaps by a body of mass different from the first, with heating measure $\alpha q$. In very rough terms this second process would be a "scaled" copy of the first, perhaps of different duration. (Note for instance that in Example 3.1 the heating measure for a process is the product of the mass of the body experiencing it and a measure that depends only on the function $\sigma : \mathscr{I} \to \Sigma$ that characterizes the process.)

We think it not unreasonable to invoke Property 4 in a general way, but we also think it to be less compelling than Properties 1–3. *If the set $\mathscr{P}_c$ does in fact enjoy Property 4 it is not difficult to see that $\mathscr{C}$, the set of cyclic heating measures, is a cone. Thus, readers willing to place Property 4 alongside Properties 1–3 as a feature to be expected of a thermodynamic theory can, everywhere in this article, replace the symbol* Cone $(\mathscr{C})$ *by $\mathscr{C}$ itself. In particular, the symbol $\hat{\mathscr{C}}$ can be interpreted as $cl(\mathscr{C})$.*

We turn now to discussion of Definition 3.2. A *Kelvin-Planck system* is a cyclic heating system which, in a certain sense, respects the Kelvin-Planck statement of the Second Law. In rough terms this version of the Second Law requires that *if, while suffering a cyclic process, a body absorbs heat from its exterior that body must also emit heat to its exterior during the process.* Whereas the First Law requires that the *net* amount of heat absorbed by a body in a cyclic process be the same as the amount of work performed by the body, the Kelvin-Planck Second Law precludes the possibility that a cyclic process might operate with perfect efficiency. That is, heat supplied to the body cannot be converted entirely into work, for there must be emission of heat from the body as well.

For a cyclic heating system $(\Sigma, \mathscr{C})$ to be a Kelvin-Planck system we shall require, among other things, that each cyclic heating measure $q \in \mathscr{C}$ have the following property: *If, for some Borel set $\Lambda \subset \Sigma$, $q(\Lambda)$ is positive, then there must exist another Borel set $\Lambda' \subset \Sigma$ such that $q(\Lambda')$ is negative.* Cast as a prohibition, this statement requires that no non-zero cyclic heating measure be a member of $\mathscr{M}_+(\Sigma)$, the set of measures that are non-negative on every Borel set in $\Sigma$.

Therefore, for a cyclic heating system $(\Sigma, \mathscr{C})$ to be a Kelvin-Planck system we shall require that

$$\mathscr{C} \cap \mathscr{M}_+(\Sigma) \text{ is at most the zero measure.} \tag{3.1}$$

For any $\alpha \in \mathbb{P}$ and any $q \in \mathscr{M}(\Sigma)$ it is clear that $\alpha q$ is a member of $\mathscr{M}_+(\Sigma)$ if and only if $q$ is. Thus, *whether or not $\mathscr{C}$ is a cone*, the condition (3.1) is equivalent to the condition

$$\text{Cone}\,(\mathscr{C}) \cap \mathscr{M}_+(\Sigma) \text{ is at most the zero measure.} \tag{3.2}$$

Although the equivalent conditions (3.1) and (3.2) imply that a cyclic heating system $(\Sigma, \mathscr{C})$ respects the Kelvin-Planck Second Law, we shall require something more of a cyclic heating system before we call it a Kelvin-Planck system.

We want our definition of a Kelvin-Planck system to carry with it the implication that cyclic processes not only obey the Second Law but also that they do not come arbitrarily close to standing in violation of it.* There are two ways in which we might try to make this precise. The first and most obvious way is to strengthen (3.1) by insisting that

$$cl(\mathscr{C}) \cap \mathscr{M}_+(\Sigma) \text{ is at most the zero measure.} \tag{3.3}$$

The second way is to strengthen (3.2) by insisting that

$$cl(\text{Cone}(\mathscr{C})) \cap \mathscr{M}_+(\Sigma) \text{ is at most the zero measure.} \tag{3.4}$$

Note that if $\mathscr{C}$ is a cone then (3.3) and (3.4) are equivalent. In light of remarks made earlier there is no distinction between (3.3) and (3.4) for readers willing to invoke Property 4 in a general way.

While conditions (3.1) and (3.2) are equivalent whether or not $\mathscr{C}$ is a cone, conditions (3.3) and (3.4) need not be equivalent if $\mathscr{C}$ is not a cone. It is only for readers who find Property 4 less than compelling that we need discuss the distinction between (3.3) and (3.4) and why we prefer one to the other. We proceed by way of a hypothetical example.

**Example 3.3.** (*A Hypothetical Two-State Substance*). Although the vector space $\mathscr{M}(\Sigma)$ in which the sets $\mathscr{C}$ and $\mathscr{M}_+(\Sigma)$ reside will generally be infinite-dimensional, it is instructive to consider the distinction between statements (3.3) and (3.4) in a model situation constructed in $\mathbb{R}^2$. For this purpose we consider a "play" substance, material points of which manifest themselves in only two states, designated $\sigma_1$ and $\sigma_2$. Moreover, we suppose that material points in state $\sigma_1$ are incapable of emitting heat and that material points in state $\sigma_2$ are incapable of absorbing heat.

Here we take $\Sigma = \{\sigma_1, \sigma_2\}$, and we identify $\mathscr{M}(\Sigma)$ with $\mathbb{R}^2$. That is, a heating measure for a process suffered by a body composed of our substance is identified with a vector $q = (q_1, q_2) \in \mathbb{R}^2$, where $q_1$ is the amount of heat absorbed by material points in state $\sigma_1$ and $-q_2$ is the amount of heat emitted by material points in state $\sigma_2$. Thus, the net amount of heat absorbed during the process is $q_1 + q_2$. If the process is cyclic this quantity is, by the First Law, identical to the net work performed by the body during the process.

For the purposes of our example we suppose that heating measures for cyclic processes admitted by our play substance are given by the set

$$\mathscr{C} = \{(m, -n) \in \mathbb{R}^2 \mid m \text{ and } n \text{ are positive integers}\},$$

and we identify $\mathscr{M}_+(\Sigma)$ with the non-negative quadrant of $\mathbb{R}^2$:

$$\mathscr{M}_+(\Sigma) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqq 0, x_2 \geqq 0\}.$$

It is easy to see that both $\mathscr{C}$ and Cone ($\mathscr{C}$) fail to meet $\mathscr{M}_+(\Sigma)$. On the other hand we have

$$cl(\mathscr{C}) = \mathscr{C} \text{ and } cl(\text{Cone}(\mathscr{C})) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqq 0, x_2 \leqq 0\}.$$

* See Remark 3.2 for further discussion of this requirement.

It is clear that $\mathscr{cl}(\mathscr{C})$ fails to meet $\mathscr{M}_+(\Sigma)$, but $\mathscr{cl}(\text{Cone}(\mathscr{C}))$ intersects $\mathscr{M}_+(\Sigma)$ along the half line $\{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \geqq 0\}$.

Thus, our model cyclic heating system $(\Sigma, \mathscr{C})$ respects condition (3.3) but not condition (3.4). The situation here might be roughly described in the following way: Even though the set $\mathscr{C}$ does not come arbitrarily close to meeting $\mathscr{M}_+(\Sigma)$, members of $\mathscr{C}$ nevertheless come arbitrarily close to being "positive in direction". This has implications for the efficiency that might be achieved in a cyclic process. By the efficiency of a cyclic process with heating measure $\varphi = (m, -n)$ we mean the net work performed by the body $(m - n)$ divided by the heat supplied to the body $(m)$. Note that

$$\sup\{(m-n)/m \mid (m, -n) \in \mathscr{C}\} = 1.$$

Therefore cyclic processes can come arbitrarily close to achieving perfect efficiency, *this despite the fact that* $\mathscr{cl}(\mathscr{C}) \cap \mathscr{M}_+(\Sigma)$, *is empty.*

However hypothetical might be the example just considered we think it makes clear in a simple way the distinction between (3.3) and (3.4). Moreover, we think the example also makes clear why statement (3.4) is to be preferred in making precise the requirement that a cyclic heating system not come arbitrarily close to reflecting a violation of the Kelvin-Planck Second Law. The statement (3.3) is by itself too weak to carry any stricture against approach to perfect efficiency in cyclic processes. Insofar as efficiency in cyclic processes is measured by the *ratio* of net heat absorption to absolute heat absorbtion, any such stricture must ultimately be cast in terms of the *direction* in $\mathscr{M}(\Sigma)$ along which cyclic heating measures might lie. In effect, the stronger statement (3.4) ensures that cyclic heating measures not come arbitrarily close to being "positive in direction".

With this in mind we take a *Kelvin-Planck system* to be a cyclic heating system that respects statement (3.4). Although we have not insisted that $\mathscr{C}$ contain the zero measure, it will nevertheless be the case that $\mathscr{cl}(\text{Cone}(\mathscr{C}))$ contains the zero measure. Since $\mathscr{M}_+(\Sigma)$ also contains the zero measure, statement (3.4) can always be written as

$$\mathscr{cl}(\text{Cone}(\mathscr{C})) \cap \mathscr{M}_+(\Sigma) = \{0\}. \tag{3.5}$$

This completes our discussion of Definition 3.2.

**Remark 3.2.** The notion of heating measure for a process, so central to what we do here, was suggested by a conversation with JAMES SERRIN in the spring of 1978. What we call heating measures were, for SERRIN, defined on a presupposed one-dimensional *hotness manifold.** That we take heating measures to be defined on a more general set of state descriptions should not obscure the fact that our use of heating measures was inspired by SERRIN'S. The idea that various versions of the Second Law could be cast in terms of heating measures is also due to SERRIN.

* More recently SERRIN'S work has been cast less in terms of heating measures than in terms of "accumulation functions" on the hotness manifold [S2, S3]. Nevertheless, his use of heating measures was explicit in a lecture published in 1977 [S1].

Whereas we require that no positive measure even be *approximated* by cyclic heating measures, our condition (3.5) reflects a somewhat stronger statement of the Kelvin-Planck Second Law than does the one SERRIN proposed (roughly, our condition (3.1)). It seems to us, however, that the stronger version merely renders explicit a presumption invoked implicitly in thermodynamic theories based on study of idealized processes. Standard textbook arguments rest heavily upon consideration of reversible processes which are approximated by real processes but which cannot themselves be realized. It is invariably presumed that such processes, fictitious though they are, should also be subject to the Second Law. In this sense the classical theories require that the Second Law be obeyed not only by real processes but also by those idealized processes the real processes approximate.

**Remark 3.3.** The reader might wish to keep in mind that a cyclic heating system is as much a description of a thermodynamical theory as it is a description of the material universe the theory purports to capture. We can imagine, for example, different theories of cyclic processes that might be suffered by homogeneous bodies of a perfect gas. In one theory the state of a material point might be described by a pair $(p, v)$, where $p$ is the pressure and $v$ is the specific volume. In another theory the state might be described by a pair $(\Theta, v)$, where $v$ is again the specific volume and $\Theta$ is an empirical temperature given, say, by a continuous monotonically increasing function of $pv$. In still another theory the state may be taken to be described *solely* by the empirical temperature.

In certain instances two superficially different theories might result in essentially identical cyclic heating systems, as when the two sets of state descriptions are identical up to homeomorphism and differences in the two sets of cyclic heating measures merely reflect the corresponding change of variable. (Consider, for example, the first two theories desribed in the preceding paragraph). In other instances two different theories of the same material universe might result in essentially distinct cyclic heating systems with somewhat different mathematical properties. For example, the sets of state descriptions for the first and third theories described in the preceding paragraph would not generally be homeomorphic.

## 4. The Existence of Clausius Temperature Scales

In this section we show that the existence of a Clausius temperature scale for a Kelvin-Planck system is an immediate consequence of the Hahn-Banach Theorem. No special features of the system need be brought into play.

**Theorem 4.1.** *Let $(\Sigma, \mathscr{C})$ be a cyclic heating system. Then the following are equivalent:*

(i) *$(\Sigma, \mathscr{C})$ is a Kelvin-Planck system*

(ii) *There exists a continuous function* $T : \Sigma \to \mathbb{P}$ *such that*

$$\int_{\Sigma} \frac{dq}{T} \leqq 0$$

*for every cyclic heating measure* $q \in \mathscr{C}$.

**Proof.** We shall prove first that (i) implies (ii). If $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system $\hat{\mathscr{C}}$ intersects $\mathscr{M}_+(\Sigma)$ only in the zero measure. In particular the closed convex cone $\hat{\mathscr{C}}$ is disjoint from the convex compact set $\mathscr{M}^1_+(\Sigma)$. Since $\mathscr{M}(\Sigma)$ is a Hausdorff locally convex topological vector space, Theorem 2.1 ensures the existence of a continuous linear functional on $\mathscr{M}(\Sigma)$ that takes positive values on $\mathscr{M}^1_+(\Sigma)$ and non-positive values on $\hat{\mathscr{C}}$. Moreover, every continuous linear functional on $\mathscr{M}(\Sigma)$ is of the kind $\tilde{\phi}$ for some $\phi \in C(\Sigma, \mathbb{R})$. (Recall Section 2.) Thus, there exists a function $\phi \in C(\Sigma, \mathbb{R})$ such that

$$\tilde{\phi}(\nu) = \int_{\Sigma} \phi \, d\nu > 0, \qquad \forall \, \nu \in \mathscr{M}^1_+(\Sigma) \tag{4.1}$$

and

$$\tilde{\phi}(\nu) = \int_{\Sigma} \phi \, d\nu \leqq 0, \qquad \forall \, \nu \in \hat{\mathscr{C}}. \tag{4.2}$$

It follows from (4.1) that $\phi(\sigma) > 0$ for every $\sigma \in \Sigma$: since the Dirac measure $\delta_\sigma$ is a member of $\mathscr{M}^1_+(\Sigma)$ we have

$$\phi(\sigma) = \int_{\Sigma} \phi \, d\delta_\sigma > 0.$$

To obtain (ii) we need only let $T : \Sigma \to \mathbb{P}$ be the reciprocal of $\phi$, invoke (4.2), and observe that $\mathscr{C}$ is contained in $\hat{\mathscr{C}}$.

Next we prove that (ii) implies (i). Let $T : \Sigma \to \mathbb{P}$ be as in (ii), let $\phi : \Sigma \to \mathbb{P}$ be the reciprocal of $T$, and let $\tilde{\phi} : \mathscr{M}(\Sigma) \to \mathbb{R}$ be the continuous linear functional induced on $\mathscr{M}(\Sigma)$ by $\phi$. That is, let

$$\tilde{\phi}(\nu) \equiv \int_{\Sigma} \phi \, d\nu = \int_{\Sigma} \frac{d\nu}{T}.$$

Since $\phi$ is positive-valued it is clear that $\tilde{\phi}(\nu)$ is positive for every $\nu \in \mathscr{M}_+(\Sigma) \setminus \{0\}$. Moreover, it follows from (ii) and the linearity of $\tilde{\phi}$ that $\tilde{\phi}(\alpha q) \leqq 0$ for every $q \in \mathscr{C}$ and every $\alpha \in \mathbb{P} \cup \{0\}$. Thus $\tilde{\phi}^{-1}(\mathbb{P})$ is open, contains $\mathscr{M}_+(\Sigma) \setminus \{0\}$, and contains no element of Cone $(\mathscr{C})$. Therefore the complement in $\mathscr{M}(\Sigma)$ of $\tilde{\phi}^{-1}(\mathbb{P})$ is closed, contains Cone $(\mathscr{C})$ (and hence its closure $\hat{\mathscr{C}}$) and does not meet $\mathscr{M}_+(\Sigma)$ except at the zero measure. Consequently, $\hat{\mathscr{C}} = cl$ (Cone $(\mathscr{C})$) meets $\mathscr{M}_+(\Sigma)$ only at the zero measure so that $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system. This completes the proof of Theorem 4.1.

Theorem 4.1 is similar in spirit to and, in fact, was motivated by SERRIN'S *Accumulation Theorem* [S2, S3]. However, the existence of temperature scales satisfying the Clausius inequality emerges in the two theorems from different premises in markedly different ways. Readers unfamiliar with SERRIN'S theorem are encouraged to see [S2] for a every readable account and, for more detail, [S3]. More general versions of the Accumulation Theorem are given in [COS] and [O].

**Definition 4.1.** *For a Kelvin-Planck system* $(\Sigma, \mathscr{C})$ *any continuous function* $T: \Sigma \to \mathrm{P}$ *such that*

$$\int_\Sigma \frac{d\mathscr{q}}{T} \leqq 0, \quad \forall\, \mathscr{q} \in \mathscr{C},$$

*will be called a* **Clausius temperature scale on** $\Sigma$. *For each* $\sigma \in \Sigma$ *the positive number* $T(\sigma)$ *will be called the* **temperature of state** $\sigma$ **on the scale** $T$. *When* $(\Sigma, \mathscr{C})$ *is a Kelvin-Planck system under study we shall denote by* $\mathscr{T}$ *the set of all Clausius temperature scales on* $\Sigma$. *That is,*

$$\mathscr{T} = \left\{ T \in C(\Sigma, \mathrm{P}) \mid \int_\Sigma \frac{d\mathscr{q}}{T} \leqq 0, \quad \forall\, \mathscr{q} \in \mathscr{C} \right\}.$$

**Remark 4.1.** Our proof of Theorem 4.1, (ii) $\Rightarrow$ (i), indicates that any Clausius temperature scale for $(\Sigma, \mathscr{C})$ will in fact satisfy the stronger requirement

$$\int_\Sigma \frac{d\mathscr{q}}{T} \leqq 0, \quad \forall\, \mathscr{q} \in \hat{\mathscr{C}}.$$

That is, $1/T$ will not only integrate non-positively against all cyclic heating measures, it will also integrate non-positively against all measures in the closure of the cone generated by the cyclic heating measures.

**Remark 4.2.** Let $T: \Sigma \to \mathrm{P}$ be any Clausius temperature scale for a Kelvin-Planck system $(\Sigma, \mathscr{C})$. If both $\mathscr{q}$ and $-\mathscr{q}$ are elements of $\hat{\mathscr{C}}$ it is an easy consequence of Remark 4.1 that

$$\int_\Sigma \frac{d\mathscr{q}}{T} = 0.$$

**Remark 4.3.** Theorem 4.1 ensures the existence of at least one Clausius temperature scale for a Kelvin-Planck system $(\Sigma, \mathscr{C})$. In general there will be many: If $T(\cdot)$ is a Clausius temperature scale then, for any $\alpha \in \mathrm{P}$, $\alpha T(\cdot)$ is another. In fact, if $T_1(\cdot)$ and $T_2(\cdot)$ are Clausius scales and $\alpha_1$ and $\alpha_2$ are any positive numbers the function $T_3: \Sigma \to \mathrm{P}$ defined by

$$\frac{1}{T_3(\cdot)} = \frac{1}{\alpha_1 T_1(\cdot)} + \frac{1}{\alpha_2 T_2(\cdot)}$$

is again a Clausius temperature scale.

Without imposition of additional restrictions upon a Kelvin-Planck system it need not be the case that all its Clausius temperature scales are positive scalar multiples of some fixed one. In Section 9 we shall provide conditions on a Kelvin-Planck system that are both necessary and sufficient to ensure that all its Clausius temperature scales are identical up to positive scalar multiplication. In the meantime we shall not impose any such requirement. Rather, we shall examine properties common to all Clausius scales of a Kelvin-Planck system, whatever they may be.

For a Kelvin-Planck system $(\Sigma, \mathscr{C})$ each Clausius scale serves to partition $\Sigma$ into *isotherms*, these being maximal sets of states that have a common temperature on that scale:

**Definition 4.2.** *Let $\mathscr{T}$ be the set of Clausius temperature scales for a Kelvin-Planck system $(\Sigma, \mathscr{C})$. For each $T \in \mathscr{T}$ and each $\sigma \in \Sigma$ the* **$T$-isotherm containing** *$\sigma$ is defined by*

$$i_T(\sigma) := \{\sigma' \in \Sigma \mid T(\sigma') = T(\sigma)\}.$$

**Remark 4.4.** In the absence of assumptions on $(\Sigma, \mathscr{C})$ it may be the case that isotherms induced in $\Sigma$ by $T \in \mathscr{T}$ are different from those induced by $T' \in \mathscr{T}$. The problem of providing a condition on a Kelvin-Planck system that is both necessary and sufficient to ensure that all its Clausius temperature scales induce identical isotherms turns out to be somewhat delicate. This problem is almost resolved by Proposition 8.2 and then is finally resolved by Corollary D.3 in Appendix D. In fact we shall have little need to deal with isotherms directly. We have introduced them here primarily so that they might be distinguished from and discussed in terms of *hotness levels*, which we regard to be more fundamental entities.

**Remark 4.5.** It is perhaps worth noting that, in the context of Examples 3.1 and 3.2, the integral in Theorem 4.1 can be cast in what some may regard to be more traditional form. In particular, if the measure $\mathscr{q}$ defined in Example 3.1 is a cyclic heating measure of a Kelvin-Planck system with Clausius scale $T: \Sigma \to \mathrm{P}$, then for the process described there we have

$$\int_{\Sigma} \frac{d\mathscr{q}}{T} = \int_{\mathscr{I}} \frac{1}{T(p(t), v(t))} [f(p(t), v(t)) \dot{p}(t) + g(p(t), v(t)) \dot{v}(t)] \, dt \leqq 0.$$

Similarly, if the measure $\mathscr{q}$ defined in Example 3.2 is a cyclic heating measure of a Kelvin-Planck system with Clausius scale $T: \Sigma \to \mathrm{P}$, then for the process considered there we have

$$\int_{\Sigma} \frac{d\mathscr{q}}{T} = \int_{\mathscr{B} \times \mathscr{I}} \frac{d\mathscr{h}}{T(\sigma(X, t))} \leqq 0.$$

## 5. A Remark on the Numerical Representation of Hotness

In the next section we shall begin our study of hotness. In particular we shall render precise for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ the idea that two states $\sigma' \in \Sigma$ and $\sigma \in \Sigma$ *are of the same hotness* (denoted $\sigma' \sim \sigma$). The relation $\sim$ will in fact be an equivalence relation that serves to partition $\Sigma$ into a set $H$ of equivalence classes called *hotness levels*. Moreover, $H$ will be given the usual quotient topology. In Section 7 we shall indicate in precise terms when $h' \in H$ is *hotter than* $h \in H$.

Underlying all the definitions we shall formulate is the view that properties of and relations within a particular thermodynamic system should be consequences of the processes that system admits. To the extent that *cyclic* processes might be used to decide when two states are of the same hotness or when one is hotter than another, these notions can be defined for a Kelvin-Planck system directly in terms of its supply of cyclic heating measures without recourse to the existence of Clausius temperature scales. Thus, the set $\mathscr{C}$ for each Kelvin-Planck system $(\Sigma, \mathscr{C})$ will serve to partition $\Sigma$ into a set $H$ of hotness levels (equipped with the quotient topology) and to determine when $h' \in H$ is hotter than $h \in H$.

Classical thermodynamics is permeated by what TRUESDELL [T2] called

> "... the silent prejudice all the pioneers of thermometry, calorimetry, and thermodynamics accepted as a matter of course: *Hotness may be represented faithfully by points on the real line.*"

In the context of this study we take such faithful representation to mean that, when $H$ is totally ordered by the *hotter than* relation given it, there exists a function $f: H \to \mathbb{R}$ that, first, provides a homeomorphism between $H$ and its image and, second, preserves order strictly:

$$h' \in H \text{ is hotter than } h \in H \Leftrightarrow f(h') > f(h).$$

With a particular notion of *hotter than* posited, we take as something *to prove* that such a function exists, that when $H$ is totally ordered it is both homeomorphic and order-similar to a subset of the real line. The fact that there is indeed something to prove has, we think, been obscured on one hand by the sheer weight of historical presumption that hotness can be reflected faithfully in a numerical scale and, on the other hand, by spurious textbook arguments purporting to show that this must be the case. We cite as an example of such an argument a few sentences from the textbook by DENBIGH [D2, p. 9], a book more carefully constructed than most:

> Now it is a fact of experience that a set of bodies can be arranged in a unique series according to their hotness, as judged by the sense of touch. That is to say, if $A$ is hotter than $B$, and $B$ is hotter than $C$, then $A$ is also hotter than $C$. The same property is shown also by the real numbers; thus if $n_a$, $n_b$ and $n_c$ are three numbers such that $n_a > n_b$ and $n_b > n_c$, then we have also $n_a > n_c$. *It follows that* the various bodies arranged in their order of hotness can each be assigned a number such that larger numbers correspond to greater degrees of hotness. The number assigned to a body may then be called its temperature, but there are obviously an infinite variety of ways in which this numbering can be carried out.

The italics are ours. Not only is it not obvious that there are an infinite variety of ways in which the numbering can be carried out, it is not obvious that there need be any. *It is not always true that the elements of a set endowed with a total order can be numbered in such a way as to reflect that order faithfully.* Although we dress the following counterexample in physical clothing, its essential mathematical content has been used elsewhere to make the same point.*

**Counterexample.** We identify the states of a hypothetical gas with ordered pairs $(p, v)$ of real numbers, where $p$ denotes the pressure and $v$ the specific volume. Moreover, we restrict our attention to the set $\Sigma$ of states such that both $p$ and $v$ lie in the interval $[1, 2]$; that is, $\Sigma = [1, 2] \times [1, 2]$. We suppose that our hypothetical gas is such that state $(p', v')$ is deemed *hotter than* state $(p, v)$ whenever $p' > p$ and, if $p' = p$, whenever $v' > v$. Thus, no two distinct states of $\Sigma$ are of the same hotness, and the hotness levels in $\Sigma$ may be identified with the elements of $\Sigma$. The *hotter than* relation described is endowed with all the properties normally associated with a total order, in particular the transitive quality upon which Denbigh rests his argument. *Nevertheless, there exists no function* $f: \Sigma \to \mathbb{R}$ *such that*

$$(p', v') \text{ is hotter than } (p, v) \Leftrightarrow f(p', v') > f(p, v).$$

**Proof.** Suppose on the contrary that such a function $f$ exists. For each $p \in [1, 2]$ let $I_p \subset \mathbb{R}$ denote the closed interval $[f(p, 1), f(p, 2)]$. If $p' > p$ then $f(p', 1) > f(p, 2)$ so that $I_{p'} \cap I_p$ is empty. Hence the map $p \in [1, 2] \mapsto I_p$ gives a bijective correspondence between numbers in the interval $[1, 2]$ and the set of disjoint closed intervals $\{I_p \mid p \in [1, 2]\}$. We have a contradiction: Points in the interval $[1, 2]$ are uncountable while the corresponding collection of disjoint closed intervals is countable (since each interval can be identified with a rational number contained within it).

Thus, to establish that the set of hotness levels for a particular system is order-similar (much less topologically similar) to a subset of the real line it is not enough merely to assert that the hotness levels are endowed with *some* total order. If such similarity is to be proved as a general feature of a class of thermodynamic systems that feature must derive from properties common to those systems and from a precisely posited notion of *hotter than*.

Here, of course, the class under study will consist of the Kelvin-Planck systems. Once specific notions of *hotness level* and *hotter than* are defined for them, each such system will be endowed with a concrete hotness structure. Thus, we shall be in a position to prove that—or at least examine conditions under which—the set of hotness levels for a Kelvin-Planck system, when totally ordered by a specified *hotter than* relation, is both homeomorphic and order-similar to a subset of the real line.

---

* In particular, it was used by Debreu [D3] in an essay beginning with these lines: "It has often been assumed in economics that if a set ... is completely ordered by the preference of some agent, it is always possible to define on that set a real-valued order-preserving function (utility, satisfaction). This is easily seen to be false."

It is natural to expect that the Clausius temperature scales for a Kelvin-Planck system should play a role in establishing such topological and order similarity. Whereas we shall regard any hotness structure for a Kelvin-Planck system as deriving directly from its supply of cyclic heating measures, we shall be left to consider the extent to which the Clausius temperature scales reflect that structure *precisely*. That is, we shall be interested in learning not only how the hotness structure of a Kelvin-Planck system restricts the nature of its Clausius temperature scales; we shall also want to know what the Clausius temperature scales tell us about its hotness structure *and, therefore, about the supply of cyclic processes from which that structure derives.*

## 6. Hotness

In this section we render precise for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ the idea that two states $\sigma' \in \Sigma$ and $\sigma \in \Sigma$ are *of the same hotness* (denoted $\sigma' \sim \sigma$). Our definition is stated solely in terms of the supply $\mathscr{C}$ of cyclic heating measures for the system at hand and does not draw upon the existence of its Clausius temperature scales. Thus, we shall be in a position to examine the extent to which the Clausius temperature scales reflect the fact that two states are of the same hotness. In particular, the equivalence relation $\sim$ will serve to partition $\Sigma$ into equivalence classes called *hotness levels*, and we can examine the relationship between these and the isotherms induced in $\Sigma$ by the individual Clausius temperature scales.

**Definition 6.1.** *For a Kelvin-Planck system* $(\Sigma, \mathscr{C})$ *we say that two states* $\sigma' \in \Sigma$ *and* $\sigma \in \Sigma$ **are of the same hotness** *(denoted* $\sigma' \sim \sigma$*) if both* $\delta_\sigma - \delta_{\sigma'}$ *and* $\delta_{\sigma'} - \delta_\sigma$ *are elements of* $\hat{\mathscr{C}}$.

We note that the definition does not require $\delta_\sigma - \delta_{\sigma'}$ and its negative to be members of $\mathscr{C}$, the set of heating measures corresponding those cyclic processes the bodies under study might admit. Rather, they are required to lie in $\hat{\mathscr{C}}$, which is to say that $\delta_\sigma - \delta_{\sigma'}$ and $\delta_{\sigma'} - \delta_\sigma$ need only be approximated by members of the cone generated by $\mathscr{C}$. In rough terms the conditions $\sigma \sim \sigma'$ might be interpreted as follows: Among the set of cyclic processes for the system under study are those in which, to good approximation, heat is absorbed [emitted] only by material points in state $\sigma'$, heat is emitted [absorbed] only by material points in state $\sigma$, and the ratio of the quantity of heat absorbed to that emitted is one.

The following theorem ensures for a Kelvin-Planck system not only that two states of the same hotness are indistinguishable on every Clausius temperature scale *but also that, if two states are not distinguished by any Clausius temperature scale, the system must be sufficiently rich in cyclic processes as to establish the two states to be of the same hotness.*

**Theorem 6.1.** *Let* $\sigma \in \Sigma$ *and* $\sigma' \in \Sigma$ *be two states of the Kelvin-Planck system* $(\Sigma, \mathscr{C})$. *Then the following are equivalent:*

(i) *$\sigma$ and $\sigma'$ are of the same hotness.*
(ii) *$T(\sigma) = T(\sigma')$ for every Clausius temperature scale $T \in \mathscr{T}$.*

Although the proof that (i) implies (ii) is immediate, the proof that (ii) implies (i) is less so. For this reason we shall find it convenient to have at our disposal two lemmas that will be used not only here but in subsequent sections as well.

**Lemma 6.1.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system, let $\nu$ be a measure in $\mathscr{M}(\Sigma)$ and let $\mathscr{K}(\nu) \subset \mathscr{M}(\Sigma)$ be the convex hull of $\{\nu\} \cup \mathscr{M}_+^1(\Sigma)$; that is, let*

$$\mathscr{K}(\nu) := \{\lambda\nu + (1-\lambda)\,\omega \mid \lambda \in [0,1], \omega \in \mathscr{M}_+^1(\Sigma)\}.$$

*If $\mathscr{K}(\nu)$ is disjoint from $\hat{\mathscr{C}}$, there exists for $(\Sigma, \mathscr{C})$ a Clausius temperature scale $T: \Sigma \to \mathrm{P}$ such that*

$$\int_\Sigma \frac{d\nu}{T} > 0.$$

**Proof.** Since $\mathscr{M}_+^1(\Sigma)$ and $\{\nu\}$ are both convex and compact, it follows that the convex hull of their union is also compact [C1, § 19.5]. Therefore, $\mathscr{K}(\nu)$ is compact, convex and, by hypothesis, disjoint from the closed convex cone $\hat{\mathscr{C}}$. Moreover, $\mathscr{K}(\nu)$ contains $\nu$ and every element of $\mathscr{M}_+^1(\Sigma)$. To obtain the desired result we need merely repeat the proof of Theorem 4.1 (i $\Rightarrow$ ii) with $\mathscr{K}(\nu)$ inserted in place of $\mathscr{M}_+^1(\Sigma)$.

**Lemma 6.2.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system for which $T$ is a Clausius temperature scale. Moreover, let $\varphi \in \mathscr{M}(\Sigma)$ be such that*

$$\int_\Sigma \frac{d\varphi}{T} = 0.$$

*If $\varphi$ is not an element of $\hat{\mathscr{C}}$ then there exists another Clausius temperature scale $T^\circ$ for $(\Sigma, \mathscr{C})$ such that*

$$\int_\Sigma \frac{d\varphi}{T^\circ} > 0.$$

**Proof.** Let $\varphi \in \mathscr{M}(\Sigma)$ satisfy the equality above, let $\lambda$ be a number in the interval $[0, 1)$, and let $\omega$ be an element of $\mathscr{M}_+^1(\Sigma)$. Then we have

$$\int_\Sigma \frac{1}{T}\, d[\lambda\varphi + (1-\lambda)\,\omega] = (1-\lambda) \int_\Sigma \frac{d\omega}{T} > 0.$$

Thus, with $\mathscr{K}(\varphi)$ as in Lemma 6.1 we can say that no element of $\mathscr{K}(\varphi)$, except perhaps for $\varphi$ itself, can be a member of $\hat{\mathscr{C}}$, for all members of $\hat{\mathscr{C}}$ integrate $1/T$ non-positively. If $\varphi$ is not contained in $\hat{\mathscr{C}}$, Lemma 6.1 ensures the existence of a Clausius scale $T^\circ$ such that the inequality in the statement of Lemma 6.2 holds.

**Proof of Theorem 6.1.** First we shall prove that (ii) implies (i). That is, we shall prove that when (ii) holds both $\delta_{\sigma'} - \delta_\sigma$ and $\delta_\sigma - \delta_{\sigma'}$ must be elements of $\hat{\mathscr{C}}$. If $T$ is a Clausius scale for $(\Sigma, \mathscr{C})$, (ii) ensures that

$$\int_\Sigma \frac{1}{T}\, d(\delta_{\sigma'} - \delta_\sigma) = \frac{1}{T(\sigma')} - \frac{1}{T(\sigma)} = 0.$$

Now if $\delta_{\sigma'} - \delta_\sigma$ is not an element of $\hat{\mathscr{C}}$, Lemma 6.2 ensures the existence of a Clausius scale $T^\circ$ such that

$$\int_\Sigma \frac{1}{T^\circ}\, d(\delta_{\sigma'} - \delta_\sigma) = \frac{1}{T^\circ(\sigma')} - \frac{1}{T^\circ(\sigma)} > 0,$$

whereupon $T^\circ(\sigma') \neq T^\circ(\sigma)$ in contradiction to (ii). Thus, $\delta_{\sigma'} - \delta_\sigma$ is an element of $\hat{\mathscr{C}}$. The proof that $\delta_\sigma - \delta_{\sigma'}$ is an element of $\hat{\mathscr{C}}$ is similar.

Finally, we prove that (i) implies (ii). If $\sigma$ and $\sigma'$ are of the same hotness then both $\delta_{\sigma'} - \delta_\sigma$ and $\delta_\sigma - \delta_{\sigma'}$ are members of $\hat{\mathscr{C}}$. If $T$ is a Clausius temperature scale for $(\Sigma, \mathscr{C})$ it is a consequence of Remark 4.2 that

$$\int_\Sigma \frac{1}{T}\, d(\delta_{\sigma'} - \delta_\sigma) = \frac{1}{T(\sigma')} - \frac{1}{T(\sigma)} = 0.$$

Therefore, $T(\sigma) = T(\sigma')$. This completes the proof of Theorem 6.1.

**Definition 6.2.** *The equivalence relation $\sim$ induces a partition of $\Sigma$ into equivalence classes called the* **hotness levels** *of the Kelvin-Planck system $(\Sigma, \mathscr{C})$. We denote by $H$ the set of hotness levels so induced in $\Sigma$ by $\mathscr{C}$, and we denote by $\pi: \Sigma \to H$ the map that assigns to each state its hotness level. Moreover, we give $H$ its quotient topology—that is, the strongest topology that renders $\pi$ continuous.*

**Remark 6.1.** Theorem 6.1 ensures that two states of different hotness will be distinguished by *some* Clausius temperature scale but not necessarily by *every* Clausius temperature scale. That is, Theorem 6.1 does not preclude the possibility that a particular Clausius temperature scale might assign the same temperature to two states of different hotness. The point here is that, without the imposition of conditions upon the Kelvin-Planck system $(\Sigma, \mathscr{C})$, *its hotness levels may be finer than the isotherms induced by some (or indeed by any) fixed choice of Clausius temperature scale.*

Nevertheless, Theorem 6.1 ensures that the hotness level containing state $\sigma$ is precisely the intersection of all $T$-isotherms containing $\sigma$ as $T$ ranges over all possible Clausius temperature scales for $(\Sigma, \mathscr{C})$:

$$\pi(\sigma) = \bigcap_{T \in \mathscr{T}} i_T(\sigma). \tag{6.1}$$

Thus, while any single Clausius temperature scale for $(\Sigma, \mathscr{C})$ might not reflect a suitably refined picture of its hotness levels, the *collection* of all possible Clausius temperature scales for $(\Sigma, \mathscr{C})$ will invariably determine its hotness levels completely.

Since, for a particular Clausius temperature scale, all states in the same hotness level take the same value of temperature, it makes sense to speak of the "temperature of a hotness level" as indicated on that scale. That is, Thorem 6.1 permits us to speak of "temperature as a function of hotness" rather than "temperature as a function of state."

**Definition 6.3.** *Let* $T : \Sigma \to \mathrm{P}$ *be a Clausius temperature scale for a Kelvin-Planck system* $(\Sigma, \mathscr{C})$ *with hotness levels H. By* $T_* : H \to \mathrm{P}$ *we mean the* **Clausius temperature scale on** $H$ **induced by** $T$ *in the following way: For* $h \in H$ *let* $\sigma \in \Sigma$ *be such that* $\pi(\sigma) = h$; *then*

$$T_*(h) = T(\sigma).$$

*The set of all Clausius temperature scales induced on H by elements of* $\mathscr{T}$ *will be denoted by* $\mathscr{T}_*$.

Definition 6.3 is summarized in the commutative diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{T} & \mathrm{P} \\ {\scriptstyle \pi}\big\downarrow & \nearrow_{T_*} & \\ H & & \end{array}$$

It will usually be clear from the context whether we have in mind a Clausius temperature scale on $\Sigma$ or a Clausius temperature scale on $H$, and we shall make no terminological distinction between the two unless clarity requires that we do so.

**Remark 6.2.** If $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system with hotness levels $H$ and if $h' \in H$ and $h \in H$ are distinct, then Theorem 6.1 and Definition 6.3 ensure the existence of $T_*^\circ \in \mathscr{T}_*$ such that $T_*^\circ(h') \neq T_*^\circ(h)$: If $\sigma'$ and $\sigma$ are states contained within $h'$ and $h$ respectively then $\sigma' \not\sim \sigma$, whereupon Theorem 6.1 ensures the existence of $T^\circ \in \mathscr{T}$ such that $T^\circ(\sigma') \neq T^\circ(\sigma)$. By taking $T_*^\circ$ to be the Clausius temperature scale on $H$ induced by $T^\circ$ we obtain the desired result.

In the following lemma we record for future use a few items of a technical nature.

**Lemma 6.3.** *Let* $(\Sigma, \mathscr{C})$ *be a Kelvin-Planck system. With H and* $\pi$ *as in Definition 6.2 and with* $\mathscr{T}_*$ *as in Definition 6.3 we have the following:*

(a) *Every* $T_* \in \mathscr{T}_*$ *is continuous.*
(b) *H is compact and Hausdorff.*
(c) $\pi : \Sigma \to H$ *is a closed mapping.*
(d) *Every hotness level, viewed as a subset of* $\Sigma$, *is compact.*
(e) *If* $\Sigma$ *has a countable base of open sets then so does H.*

**Proof.** (a) The proof amounts to a straightforward application of Proposition 6, p. 39 of [B].

(b) That $H$ is compact follows from the fact that it is the image of the compact space $\Sigma$ under the continuous mapping $\pi$. That $H$ is Hausdorff may be seen as follows: Let $h$ and $h'$ be distinct elements of $H$. Then Remark 6.2 ensures the existence of $T_*^\circ \in \mathcal{T}_*$ such that $T_*^\circ(h') \neq T_*^\circ(h)$. If $I' \subset \mathbb{P}$ and $I \subset \mathbb{P}$ are disjoint open intervals containing $T_*^\circ(h')$ and $T_*^\circ(h)$ respectively, then the continuity of $T_*^\circ$ ensures that $(T_*^\circ)^{-1}(I')$ and $(T_*^\circ)^{-1}(I)$ are disjoint open subsets of $H$ containing $h'$ and $h$ respectively.

(c) Let $F \subset \Sigma$ be a closed (and therefore compact) subset of the compact set $\Sigma$. Since $\pi$ is continuous $\pi(F)$ is a compact subset of the Hausdorff space $H$ and must therefore be closed.

(d) Let $h \in H$ be a hotness level. Since $H$ is Hausdorff the singleton $\{h\}$, viewed as a subset of $H$, is closed. Consequently, the continuity of $\pi$ ensures that $\pi^{-1}(h)$ is a closed and therefore compact subset of the compact set $\Sigma$.*

(e) Part (e) is a consequence of parts (c) and (d) and two theorems in [K]: Theorem 12, p. 99, and Theorem 20, p. 148. This completes the proof of Lemma 6.3.

**Remark 6.3.** Although what we say here will play no explicit role in subsequent sections, we wish to point out (without proof) that every Kelvin-Planck system $(\Sigma, \mathscr{C})$ affords a formulation in which its set $\Sigma$ of state descriptions is entirely replaced by its set $H$ of hotness levels. First we note that every signed Borel measure on $\Sigma$, say $v \in \mathscr{M}(\Sigma)$, induces a signed Borel measure on $H$, $v_* \in \mathscr{M}(H)$, as follows: For each Borel set $B \subset H$ let

$$v_*(B) = v(\pi^{-1}(B)).$$

If $\mathscr{C}_* \subset \mathscr{M}(H)$ is the set of measures induced in this way by the cyclic heating measures $\mathscr{C} \subset \mathscr{M}(\Sigma)$, then $(H, \mathscr{C}_*)$ is again a Kelvin-Planck system. Moreover, the set of Clausius temperature scales for $(H, \mathscr{C}_*)$ is precisely the set $\mathcal{T}_*$ induced in the sense of Definition 6.3 by the set $\mathcal{T}$ of Clausius temperature scales for $(\Sigma, \mathscr{C})$.

## 7. The Ordering of Hotness Levels

Having described the set $H$ of hotness levels for a Kelvin-Planck system $(\Sigma, \mathscr{C})$, we now wish to give meaning to the idea that $h' \in H$ is *hotter than* $h \in H$. In fact, we shall examine several slightly different definitions of *hotter than*, these having somewhat different consequences. In the spirit of the philosophy espoused earlier, each definition is stated solely in terms of the supply $\mathscr{C}$ of cyclic heating measures for the system at hand—or, more precisely, in terms of the presence of

* A somewhat different proof of (d) proceeds as follows: It is an easy consequence of Definition 4.2 that every isotherm is closed. Thus, (6.1) ensures that every hotness level is the intersection of closed sets and is therefore a closed subset of the compact set $\Sigma$.

certain elements in the related set $\hat{\mathscr{C}}$. Thus, the definitions will reflect the perspective of an observer who, in trying to decide whether "$h'$ is hotter than $h$", might know nothing more than the cyclic process admissible within a particular thermodynamic theory under study. The viewpoint we take here is close in spirit, but not in detail, to that taken by TRUESDELL [T2] and later by PITTERI [P], both within a more classical setting.

Because the various notions of *hotter than* for a Kelvin-Planck system will be defined without reference to its Clausius temperature scales, we shall be left to examine how each *hotter than* relation is reflected in those scales. It is worth emphasizing once again that we shall not be content to ask what implications a statement like "$h'$ is hotter than $h$" has for the relative temperatures of hotness levels $h'$ and $h$ on the Clausius scales. We shall also want to know, conversely, when one can infer that "$h'$ is hotter than $h$" from the temperatures the Clausius scales assign to $h'$ and $h$. This last question is the more difficult: Since our definition of *hotter than* will be posed in terms of the supply of cyclic heating measures, an affirmative answer will require that we prove, in effect, that the Kelvin-Planck system under study is suitably rich in cyclic processes.

In all there will be four definitions of *hotter than* we shall wish to study. These give increasingly stronger partial orders to the hotness levels, and the four are numbered in such a way as to reflect their increasing strength. That is, the weakest is termed *hotter than in the first sense* (denoted ${}_1\!\succ$) while the strongest is termed *hotter than in the fourth sense* (denoted ${}_4\!\succ$). Of the four relations only the first, third, and fourth will be examined fully in this section.

Our definition of *hotter than in the second sense* is somewhat more technical than the others, and for this reason we place its statement and examine its consequences in Appendix D. Our feeling that this more technical definition should be included at all derives from the fact that it makes possible complete answers to certain natural questions, an example of which is the following: *Under what circumstances will a Kelvin-Planck system $(\Sigma, \mathscr{C})$ have the property that its hotness levels (viewed as subsets of $\Sigma$) are identical to the isotherms induced in $\Sigma$ by every Clausius temperature scale on $\Sigma$?*

For many purposes the relation ${}_2\!\succ$ is the "right" notion of *hotter than*, and it is unfortunate that its definition is somewhat less straightforward than the others. However, the slightly stronger relation ${}_3\!\succ$ is very close to ${}_2\!\succ$ and provides a reasonable if imperfect substitute for it. In this section and the next, therefore, we have employed the relation ${}_3\!\succ$ where ${}_2\!\succ$ would have been better, thereby sacrificing sharper results for the sake of clarity. To compensate for this we have indicated in various places what improvements can be made so that interested readers might pursue them in Appendix D.

We turn now to our first notion of *hotter than.*

**Definition 7.1.** *For a Kelvin-Planck system $(\Sigma, \mathscr{C})$ with hotness levels $H$ we say that* $h' \in H$ **is hotter than** $h \in H$ **in the first sense** (*denoted* $h' {}_1\!\succ h$) *if* $h' \neq h$ *and there exists* $\varphi \in \hat{\mathscr{C}}$ *of the form*

$$\varphi = \mu' - \mu + \nu,$$

*where $\nu$, $\mu'$, $\mu$ are members of $\mathcal{M}_+(\Sigma)$ and the measures $\mu'$ and $\mu$ are such that* $\operatorname{supp}\mu' \subset h'$, $\operatorname{supp}\mu \subset h$ *and* $\mu'(h') = \mu(h) > 0$. *(Here $\nu$ may be the zero measure.)*

Viewed as subsets of $\Sigma$, the hotness levels $h'$ and $h$ are compact (Lemma 6.3) and are therefore Borel sets. If, in Definition 7.1, $\varphi$ were to be interpreted as the heating measure for a body undergoing a cyclic process,* the number $\varphi(h')$would be the net amount of heat *absorbed* during the process by material points in states of hotness $h'$, the number $-\varphi(h)$ would be the net amount of heat *emitted* by material points in states of hotness $h$, and the number $\varphi(\Sigma)$ would be the net amount of heat absorbed by all material points during the process. Since $\mu'$ and $\mu$ are required to have support in $h'$ and $h$, respectively, it is not difficult to see that

$$\varphi(h') = \mu'(h') + \nu(h') > 0,$$

$$-\varphi(h) = \mu(h) - \nu(h),$$

and

$$\varphi(\Sigma) = \mu'(h') - \mu(h) + \nu(\Sigma) = \nu(\Sigma) \geqq 0.$$

Moreover, $\varphi$ cannot be negative on any Borel set that fails to meet $h$.

**Remark 7.1.** It is important to note that in Definition 7.1 we place no explicit restriction upon the measure $\nu$ other than it be a member of $\mathcal{M}_+(\Sigma)$, possibly the zero measure. In particular, its support might or might not meet $h$ or $h'$. There is, however, an implicit restriction upon $\nu$. Since $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system the measure $\varphi \in \hat{\mathscr{C}}$ described in the definition must have a negative part. Because $\varphi$ can be negative only on Borel sets that meet $h$ we must have that $\nu - \mu$ is negative on some Borel set of states contained within the hotness level $h$.

**Remark 7.2.** Keeping in mind that $\hat{\mathscr{C}} = \mathscr{cl}\,[\text{Cone}\,(\mathscr{C})]$, we might interpret Definition 7.1 in rough terms as follows: We say that $h' \,{}_1\!\succ h$ if among the cyclic processes for the system under study are those which, to good approximation, have the following qualities:

(a) There is heat emission only from material points in states of hotness $h$.
(b) There may be heat absorption by material points in states of any hotness other than $h$, but the ratio of the amount of heat absorbed by material points in states of hotness $h'$ to the amount emitted by material points in states of hotness $h$ is *at least one.*
(c) The net amount of heat absorbed during the course of the process is *non-negative.*

The third property is a consequence of the others. We choose to make it explicit primarily so that it may be drawn upon in discussions of the distinction between Definition 7.1 and stronger definitions of *hotter than* we shall also examine. Although the First Law of Thermodynamics plays no formal role in this study,

* Note that Definition 7.1 does not require that $\varphi$ be the heating measure for some cyclic process, merely that it lie in the closure of the cone generated by such measures.

we note in passing that it requires of cyclic processes enjoying the third property that work done upon the body undergoing the process be *non-positive*.

**Theorem 7.1.** *Let h and h' be hotness levels for the Kelvin-Planck system* $(\Sigma, \mathscr{C})$. *Then the following are equivalent*:

(i) $h' \,_1\!\!> h$.

(ii) *For every Clausius temperature scale* $T_* \in \mathscr{T}_*$ *it is true that* $T_*(h') \geqq T_*(h)$, *and there exists at least one Clausius temperature scale* $T_*^\circ \in \mathscr{T}_*$ *for which* $T_*^\circ(h') > T_*^\circ(h)$.

**Proof.** To prove that (i) implies (ii) we suppose that $\varphi \in \hat{\mathscr{C}}$ is a measure of the kind described in Definition 7.1. Furthermore, we suppose that $T$ is a Clausius temperature scale on $\Sigma$ and that $T_*$ is the Clausius temperature scale induced on $H$ by $T$. From Remark 4.1 we have

$$\begin{aligned} 0 &\geqq \int_\Sigma \frac{d\varphi}{T} \\ &= \int_\Sigma \frac{1}{T}\, d(\mu' - \mu + \nu) \\ &= \int_{h'} \frac{d\mu'}{T} - \int_h \frac{d\mu}{T} + \int_\Sigma \frac{d\nu}{T} \\ &= \frac{\mu'(h')}{T_*(h')} - \frac{\mu(h)}{T_*(h)} + \int_\Sigma \frac{d\nu}{T} \\ &\geqq \mu'(h')\left(\frac{1}{T_*(h')} - \frac{1}{T_*(h)}\right). \end{aligned} \tag{7.1}$$

The last estimate in (7.1) follows from the fact that $\mu'(h') = \mu(h)$ and the requirement that $\nu$ be a member of $\mathscr{M}_+(\Sigma)$, possibly the zero measure. Since $\mu'(h')$ is positive (7.1) implies that

$$T_*(h') \geqq T_*(h), \quad \forall\, T_* \in \mathscr{T}_*.$$

That equality cannot hold for all $T_* \in \mathscr{T}_*$ follows from Remark 6.2.

Next we shall prove that (ii) implies (i). We must show that when (ii) holds there exists in $\hat{\mathscr{C}}$ a measure of the kind described in Definition 7.1. In fact, we shall prove that for any two states $\sigma' \in h'$ and $\sigma \in h$ there must exist in $\hat{\mathscr{C}}$ a measure of the form

$$\lambda(\delta_{\sigma'} - \delta_\sigma) + (1 - \lambda)\,\omega, \quad \lambda \in (0, 1], \quad \omega \in \mathscr{M}_+^1(\Sigma). \tag{7.2}$$

Any such measure will clearly satisfy the requirements of Definition 7.1. Suppose on the contrary that there exists no such measure in $\hat{\mathscr{C}}$. Since $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system, neither can $\hat{\mathscr{C}}$ contain a measure of $\mathscr{M}_+^1(\Sigma)$. Consequently, the set

$\mathscr{K}(\delta_{\sigma'} - \delta_\sigma)$ described in Lemma 6.1 must be disjoint from $\hat{\mathscr{C}}$, whereupon that lemma ensures the existence of a Clausius temperature scale $T \in \mathscr{T}$ such that

$$\int_{\Sigma} \frac{1}{T}\, d(\delta_{\sigma'} - \delta_\sigma) = \frac{1}{T(\sigma')} - \frac{1}{T(\sigma)} > 0. \tag{7.3}$$

From (7.3) it follows that $T(\sigma) > T(\sigma')$. If $T_* \in \mathscr{T}$ is the Clausius temperature scale on $H$ induced by $T$, then $T_*(h) > T_*(h')$. But this contradicts (ii).

**Remark 7.3.** From our proof of Theorem 7.1 we can infer that the statements (i) and (ii) are equivalent to yet another statement:

(iii) *For any pair of states $\sigma' \in h'$ and $\sigma \in h$ there exists $\nu \in \mathscr{M}_+(\Sigma)$, possibly the zero measure, such that $\delta_{\sigma'} - \delta_\sigma + \nu \in \hat{\mathscr{C}}$.*

It is clear that (iii) implies (i). To see that (ii) implies (iii) we need only recall that (ii) implies the existence in $\hat{\mathscr{C}}$ of a measure of the kind described in (7.2). Keeping in mind that $\hat{\mathscr{C}}$ is a cone, we can multiply such a measure by $1/\lambda$ to obtain the desired result.

**Remark 7.4.** Definition 7.1 does not by itself make obvious the fact that the relation ${}_1\succ$ provides a partial order on the set of hotness levels of a Kelvin-Planck system. On the other hand the equivalence of statements (i) and (ii) in Theorem 7.1 make the antisymmetry and transitivity of readily apparent.

Theorem 7.1 asserts that, for each *fixed* pair of hotness levels $h'$ and $h$ such that $h' \,{}_1\!\succ h$, there will exist some $T_*^\circ \in \mathscr{T}_*$ for which $T_*^\circ(h') > T_*^\circ(h)$. The theorem does not assert the existence of a *single* $T_*^\circ \in \mathscr{T}_*$ that will work in this way for *every* pair of hotness levels that are comparable with respect to the relation ${}_1\succ$. If, however, $\Sigma$ is not merely compact and Hausdorff but also a metric space, then the existence of such a temperature scale is ensured.* This is the substance of our next theorem. Its proof is presented in Appendix B.

**Theorem 7.2.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system with hotness levels $H$. If $H$ has a countable base of open sets (or, equivalently, is metrizable) then there exists a Clausius temperature scale $T_*^\circ \in \mathscr{T}_*$ with the following property: $T_*^\circ(h') > T_*^\circ(h)$ for every pair $h' \in H$, $h \in H$ such that $h' \,{}_1\!\succ h$. In particular, such a temperature scale exists if $\Sigma$ is a metric space.*

As was mentioned earlier our definition of *hotter than in the second sense* is stated and discussed in Appendix D. Therefore, we turn to our definition of *hotter than in the third sense.*

**Definition 7.2.** *For a Kelvin-Planck system $(\Sigma, \mathscr{C})$ with hotness levels $H$ we say that* $h' \in H$ **is hotter than** $h \in H$ **in the third sense** *(denoted $h' \,{}_3\!\succ h$) if*

* We are grateful to PAUL BERNER for suggesting that this might be the case.

*$h' \neq h$ and there exists $\varphi \in \hat{\mathscr{C}}$ of the kind described in Definition 7.1 with $\nu \neq 0$.*

**Remark 7.5.** Clearly, $h'\ {}_3\!\succ h$ implies that $h'\ {}_1\!\succ h$. In fact, to interpret the relation ${}_3\!\succ$ we need only repeat our interpretation of ${}_1\!\succ$ (Remark 7.2) with items (a) and (b) left intact but with item (c) modified to indicate that the net amount of heat absorbed during the course of the process is *positive* (since $\varphi(\Sigma) = \nu(\Sigma) > 0$). The First Law requires of cyclic processes having this quality that work done upon the body undergoing the process be *negative*, which is to say that the body does work.

**Theorem 7.3.** *Let h and h′ be distinct hotness levels of the Kelvin-Planck system $(\Sigma, \mathscr{C})$, and suppose that h and h′ are comparable with respect to the relation ${}_3\!\succ$. Then the following are equivalent:*

(i) $h'\ {}_3\!\succ h$.

(ii) $T_*(h') > T_*(h)$ *for every Clausius temperature scale* $T_* \in \mathscr{T}_*$.

**Proof.** To prove that (i) implies (ii) we need only repeat the analogous argument for Theorem 7.1 with $\nu \in \mathscr{M}_+(\Sigma)$ presumed non-zero; then inequality must obtain in the last estimate of (7.1), and statement (ii) above follows as a result.

To prove that (ii) implies (i) we draw upon the supposition that $h$ and $h'$ are comparable with respect to ${}_3\!\succ$—that is, that either $h'\ {}_3\!\succ h$ or $h\ {}_3\!\succ h'$. If the latter holds true then our proof that (i) implies (ii) would ensure that $T_*(h) > T_*(h')$ for every $T_* \in \mathscr{T}_*$. But this stands in contradiction to (ii).

**Remark 7.6.** Apart from their apparent differences there is a distinction between Theorems 7.1 and 7.3 that is easy to overlook. The implication (ii) $\Rightarrow$ (i) of Theorem 7.1 *does not require the prior supposition that h′ and h be ${}_1\!\succ$-comparable*; indeed it was proved that statement (ii) by itself ensures that $h'$ and $h$ are ${}_1\!\succ$-comparable with $h'\ {}_1\!\succ h$. On the other hand, the opening sentence of Theorem 7.3 *presupposes* that $h'$ and $h$ are ${}_3\!\succ$-comparable. Although statement (ii) of Theorem 7.3 implies that $h'\ {}_1\!\succ h$ (by virtue of Theorem 7.1), it does not by itself ensure that $h'$ and $h$ are ${}_3\!\succ$-comparable. We present a counterexample in Appendix C.

In rough terms the situation might be described as follows: If, for a Kelvin-Planck system, every Clausius scale assigns a higher temperature to hotness level $h'$ than it does to hotness level $h$ the system must be sufficiently rich in cyclic processes as to establish that $h'$ is hotter than $h$ in the first sense; however, the system may be insufficiently rich in cyclic processes to ensure that $h'$ is hotter than $h$ in the third sense. As we shall see in Remark 7.9 this state of affairs can obtain only in special circumstances.

We should indicate here that the notion or *hotter than in the second sense* (discussed in Appendix D) will be such that the following equivalence obtains:

$$h'\ {}_2\!\succ h \Leftrightarrow T_*(h') > T_*(h), \quad \forall\, T_* \in \mathscr{T}_*.$$

In particular, it will be the case that, whenever every Clausius scale on $H$ assigns a higher temperature to $h'$ than it does to $h$, then $h'$ and $h$ *must* be ${}_2\succ$-comparable.

**Remark 7.7.** We would like to show that for a Kelvin-Planck system the relation ${}_3\succ$, like the relation ${}_1\succ$, gives a partial ordering of the set of hotness levels. Remark 7.6 suggests that, in so doing, we must take care in drawing upon the equivalence of statements (i) and (ii) of Theorem 7.3. Proof that ${}_3\succ$ is antisymmetric provides no difficulty; for if $h' \neq h$, $h' \,{}_3\!\succ h$ and $h \,{}_3\!\succ h'$, the implication (i) $\Rightarrow$ (ii) of Theorem 7.3 gives an immediate contradiction. Proof of transitivity is somewhat more delicate. If $h'' \,{}_3\!\succ h'$ and $h' \,{}_3\!\succ h$, then we may deduce from the implicatlon (i) $\Rightarrow$ (ii) of Theorem 7.3 that $T_*(h'') > T_*(h)$ for every $T_* \in \mathscr{T}_*$. But this by itself does not imply that $h''$ and $h$ are ${}_3\succ$-comparable. Consequently, proof that the relation ${}_3\succ$ is transitive must appeal more directly to Definition 7.2. In fact, we present such a proof in Appendix C.

We turn finally to our fourth notion of *hotter than.*

**Definition 7.3.** *For a Kelvin-Planck system* $(\Sigma, \mathscr{C})$ *with hotness levels* $H$ *we say that* $\boldsymbol{h' \in H}$ **is hotter than** $\boldsymbol{h \in H}$ **in the fourth sense** (*denoted* $h' \,{}_4\!\succ h$) *if* $h' \neq h$ *and there exists* $\varphi \in \hat{\mathscr{C}}$ *of the kind described in Definition 7.1 with* $\nu(h') > 0$.

**Remark 7.8.** Clearly we have the implications

$$h' \,{}_4\!\succ h \;\Rightarrow\; h' \,{}_3\!\succ h \;\Rightarrow\; h' \,{}_1\!\succ h.$$

To interpret Definition 7.3 we can repeat Remark 7.2 with (a) left intact, with (c) modified (as in Remark 7.5) to indicate that the net amount of heat absorbed is *positive*, but now with (b) also modified to indicate that the ratio of the quantity of heat absorbed at hotness $h'$ to that emitted at hotness $h$ *exceeds one.*

**Theorem 7.4.** *Let* $h'$ *and* $h$ *be hotness levels for a Kelvin-Planck system* $(\Sigma, \mathscr{C})$. *Then the following are equivalent*:

(i) $h' \,{}_4\!\succ h$.

(ii) *There exists* $\alpha > 0$ *such that, for every Clausius temperature scale* $T_* \in \mathscr{T}_*$,

$$\frac{T_*(h')}{T_*(h)} > 1 + \alpha.$$

**Proof.** First we shall prove that (i) implies (ii). Let $T$ be a Clausius temperature scale on $\Sigma$, and let $T_*$ be the Clausius temperature scale induced on $H$ by $T$. For any $\nu \in \mathscr{M}_+(\Sigma)$ we have

$$\int_\Sigma \frac{d\nu}{T} \geqq \int_{h'} \frac{d\nu}{T} = \frac{\nu(h')}{T_*(h')}. \tag{7.4}$$

With $\mathscr{q}$ as in Definition 7.1, we may use (7.4) to improve our last estimate in (7.1). In this way we obtain the inequality

$$0 \geqq \mu'(h')\left(\frac{1}{T_*(h')} - \frac{1}{T_*(h)}\right) + \frac{\nu(h')}{T_*(h')} \tag{7.5}$$

or, equivalently,

$$\frac{T_*(h')}{T_*(h)} \geqq 1 + \frac{\nu(h')}{\mu'(h')}. \tag{7.6}$$

Thus if, as in Definition 7.3, $\nu(h')$ is positive, we may obtain the desired result by taking $\alpha$ to be any positive number less than $\nu(h')/\mu'(h')$.

To prove that (ii) implies (i) we must show that, when (ii) holds, $\hat{\mathscr{C}}$ must contain a measure of the kind described in Definition 7.1 with $\nu(h') > 0$. In fact, we shall show that if (ii) holds and if $\sigma'$ and $\sigma$ are any states contained with $h'$ and $h$, respectively, then $\hat{\mathscr{C}}$ must contain an element of the form

$$\lambda[(1+\alpha)\,\delta_{\sigma'} - \delta_\sigma] + (1-\lambda)\,\omega, \quad \lambda \in (0,1], \quad \omega \in \mathscr{M}^1_+(\Sigma). \tag{7.7}$$

(To see that such a measure satisfies the requirements of Definition 7.3 we may take $\mu' = \lambda\delta_{\sigma'}$, $\mu = \lambda\,\delta_\sigma$ and $\nu = \lambda\alpha\delta_{\sigma'} + (1-\lambda)\,\omega$.) Suppose on the contrary that $\hat{\mathscr{C}}$ contains no such measure. Neither can $\hat{\mathscr{C}}$ contain a measure of $\mathscr{M}^1_+(\Sigma)$, for $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system. Consequently, the set $\mathscr{K}[(1+\alpha)\,\delta_{\sigma'} - \delta_\sigma]$ described in Lemma 6.1 must be disjoint from $\hat{\mathscr{C}}$, in which case that lemma ensures the existence of a Clausius temperature scale $T \in \mathscr{T}$ such that

$$\int_\Sigma \frac{1}{T}\, d[(1+\alpha)\,\delta_{\sigma'} - \delta_\sigma] = \frac{(1+\alpha)}{T(\sigma')} - \frac{1}{T(\sigma)} > 0. \tag{7.8}$$

For this scale, then, we have

$$\frac{T(\sigma')}{T(\sigma)} < 1 + \alpha. \tag{7.9}$$

If $T_*$ is the Clausius temperature scale induced on $H$ by $T$ we obtain from (7.9) the inequality

$$\frac{T_*(h')}{T_*(h)} < 1 + \alpha,$$

which contradicts (ii).

**Remark 7.9.** We note that the implication (ii) ⇒ (i) of Theorem 7.4 *does not require the prior supposition that the hotness levels $h'$ and $h$ be ${}_4\succ$-comparable.* Roughly speaking, then, the collection of Clausius temperature scales for a Kelvin-Planck system can have property (ii) *only if the system is sufficiently rich in cyclic processes as to establish that $h'$ is hotter than $h$ in the fourth sense.*

Thus, Theorem 7.4 is similar in character to Theorem 7.1 but dissimilar in character to Theorem 7.3 (Remark 7.6). In fact, Theorem 7.4 permits us to flesh out Remark 7.6 a little further. It tells us that property (ii) of Theorem 7.3 can fail to ensure the ${}_3\succ$-comparability of hotness levels $h'$ and $h$ only under special circumstances: As $T_*$ ranges over all possible Clausius temperature scales on $H$ the ratio $T_*(h')/T_*(h)$ must take values arbitrarily close to one. For if that ratio is bounded away from one, Theorem 7.4 requires that $h' \ {}_4\succ h$, and this in turn implies that $h' \ {}_3\succ h$.

**Remark 7.10.** The proof of Theorem 7.4 ensures that statements (i) and (ii) are equivalent to:

(iii) *For any states* $\sigma' \in h'$ *and* $\sigma \in h$ *there exists in* $\hat{\mathscr{C}}$ *an element of the form* $\delta_{\sigma'} - \delta_\sigma + \nu$, *where* $\nu(h') > 0$.

Proof of this assertion is similar to that offered in Remark 7.3.

**Remark 7.11.** That ${}_4\succ$ gives a partial order to the hotness levels of a Kelvin-Planck system is an easy consequence of the equivalence of (i) and (ii) in Theorem 7.4. Antisymmetry of ${}_4\succ$ is immediate. To prove transitivity we suppose that $h'' \ {}_4\succ h'$ and that $h' \ {}_4\succ h$. Theorem 7.4 ensures the existence of positive numbers $\alpha_1$ and $\alpha_2$ such that, for every $T_* \in \mathscr{T}_*$,

$$\frac{T_*(h'')}{T_*(h')} > 1 + \alpha_1 \text{ and } \frac{T_*(h')}{T_*(h)} > 1 + \alpha_2 .$$

Thus, we have

$$\frac{T_*(h'')}{T_*(h)} > (1 + \alpha_1)(1 + \alpha_2) = 1 + \alpha_1 + \alpha_2 + \alpha_1\alpha_2$$

for every $T_* \in \mathscr{T}_*$, whereupon Theorem 7.4 ensures that $h'' \ {}_4\succ h$.

## 8. Totally Ordered Kelvin-Planck Systems: The One-Dimensionality of the Hotness Set

We are now in a position to take up questions raised in Section 5. In particular, when the hotness levels of a Kelvin-Planck system are *totally* ordered by a suitable notion of *hotter than* we wish to examine conditions under which the hotness levels, in some sense, constitute a "one-dimensional hotness manifold" and under which the order can be faithfully reflected in a real numerical scale. We made the point in Section 5 that neither of these properties is ensured by the mere presumption that the hotness levels are totally ordered by *some* relation. Rather, they must derive from qualities peculiar to the class of systems under study taken together with features of the particular order posited.

Here we shall draw upon qualities common to Kelvin-Planck systems to prove that *if the set H of hotness levels is totally ordered by* ${}_3\succ$ *then H must be homeomorphic to a subset of the real line and the order must be faithfully reflected in a Clausius temperature scale.* As we shall see in Appendix D this also holds true when $H$ is totally ordered by the weaker relation ${}_2\succ$. In fact we shall show in this section that *the same result obtains when H is totally ordered by the still weaker relation* $_1\succ$, *provided the underlying set of states* $\Sigma$ *is a metric space*. It should not detract from their importance that these assertions are easy corollaries of theorems already stated.

In general the *hotter than* relations defined earlier will merely lend a *partial* order to the hotness levels of a Kelvin-Planck system, and we do not take as a postulate that even the weakest of these ($_1\succ$) must necessarily provide a *total* order for any system likely to command our attention. Rather, it is our purpose in this section to examine consequences of total ordering when it obtains. Before undertaking this examination, however, we think it worthwhile to point out that if two hotness levels *fail* to be comparable even with respect to the weakest *hotter than* relation there will exist "conflicting" Clausius temperature scales:

**Proposition 8.1.** *Let* $(\Sigma, \mathscr{C})$ *be a Kelvin-Planck system with hotness levels* $H$. *If* $h' \in H$ *and* $h \in H$ *are distinct and are not comparable with respect to the relation* $_1\succ$ *then there exist Clausius temperature scales* $T^1_*$, $T^2_*$, *and* $T^3_*$ *on* $H$ *such that*

(i) $T^1_*(h') > T^1_*(h)$,

(ii) $T^2_*(h') < T^2_*(h)$,

*and* (iii) $T^3_*(h') = T^3_*(h)$.

**Proof.** To prove (i) we merely note that if $T_*(h') \leqq T_*(h)$ for all $T_* \in \mathscr{T}_*$ then Remark 6.2. and Theorem 7.1 require that $h\ _1\!\succ h'$, in contradiction to the hypothesis. The proof of (ii) is similar. To prove (iii) we let $T^1_*$ and $T^2_*$ be as in (i) and (ii), and we let $T^1$ and $T^2$ be the Clausius temperature scales on $\Sigma$ defined by $T^1 = T^1_* \circ \pi$ and $T^2 = T^2_* \circ \pi$, where $\pi : \Sigma \to H$ is as in Definition 6.2. By virtue of Remark 4.3 the function $T^3 : \Sigma \to \mathbb{P}$ defined by

$$\frac{1}{T^3(\cdot)} = \frac{T^1_*(h') - T^1_*(h)}{T^1_*(h')\, T^1_*(h)}\, \frac{1}{T^2(\cdot)} + \frac{T^2_*(h) - T^2_*(h')}{T^2_*(h')\, T^2_*(h)}\, \frac{1}{T^1(\cdot)}$$

is again a Clausius temperature scale on $\Sigma$. Moreover, it is easy to confirm that $T^3(\sigma') = T^3(\sigma)$ for any $\sigma' \in h'$ and any $\sigma \in h$. Thus, the Clausius temperature scale $T^3_*$ induced on $H$ by $T^3$ has the property that $T^3_*(h') = T^3_*(h)$.

In Remark 6.1 we made the point that for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ the hotness levels, viewed as subsets of $\Sigma$, might be finer than the isotherms induced in $\Sigma$ by a particular Clausius temperature scale. For each Clausius scale it will always be the case that every hotness level is entirely contained within an isotherm, but a particular Clausius scale might induce in $\Sigma$ an isotherm which is the union of two or more hotness levels. Consequently, we are compelled to ask when it will be the case that, for *every* Clausius temperature scale, the induced isotherms coincide with the hotness levels. For this to happen it is necessary that the hotness levels be totally ordered by $_1\succ$ and sufficient that they be totally ordered by $_3\succ$:

**Proposition 8.2.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system, let $H$ be its set of hotness levels, and let $\mathscr{T}$ be its collection of Clausius temperature scales. Among the following statements we have the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(i) *$H$ is totally ordered by ${}_3\succ$.*

(ii) *For each $T \in \mathscr{T}$, every $T$-isotherm coincides with a hotness level. That is, $i_T(\sigma) = \pi(\sigma)$ for every $T \in \mathscr{T}$ and every $\sigma \in \Sigma$.*

(iii) *$H$ is totally ordered by ${}_1\succ$.*

**Proof.** First we prove that (i) implies (ii). Suppose on the contrary that (i) obtains but that there exist states $\sigma'$ and $\sigma$ belonging to distinct hotness levels $h'$ and $h$ such that $T(\sigma') = T(\sigma)$ for some $T \in \mathscr{T}$. Let $T_*$ be the Clausius scale induced on $H$ by $T$. Because $h'$ and $h$ are distinct and ${}_3\succ$-comparable we have, by Theorem 7.3, that $T_*(h') \neq T_*(h)$. But we also have that $T(\sigma') = T_*(h')$ and $T(\sigma) = T_*(h)$, which give a contradiction.

Next we will suppose that (ii) holds but that (iii) does not. If (iii) does not hold then there exist distinct hotness levels $h'$ and $h$ that are not ${}_1\succ$-comparable. Thus Proposition 8.1 ensures the existence of a Clausius temperature scale $T_*$ on $H$ such that $T_*(h') = T_*(h)$. Let $T$ be the Clausius scale on $\Sigma$ defined by $T(\cdot) = T_* \circ \pi(\cdot)$. The distinct hotness levels $h'$ and $h$ clearly reside within the same $T$-isotherm in contradiction to (ii).

**Remark 8.1.** Neither of the implications in Proposition 8.2 can be reversed. The counterexample presented in Appendix C for a somewhat different purpose also serves to indicate that, for statements (i) and (ii) of Proposition 8.2, the assertion (ii) $\Rightarrow$ (i) is false. It is not difficult to construct counterexamples to the assertion (iii) $\Rightarrow$ (ii). In fact we shall prove in Appendix D that statement (ii) of Proposition 8.2 is *equivalent* to the following statement: *$H$ is totally ordered by ${}_2\succ$.*

We turn now to the main theorems of this section.

**Theorem 8.1.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system with hotness levels $H$. If $H$ is totally ordered by ${}_1\succ$ and $\Sigma$ is a metric space then there exists a Clausius temperature scale $T_*^0 \in \mathscr{T}_*$ that encodes the order precisely*; *that is,*

$$T_*^0(h') > T_*^0(h) \Leftrightarrow h' \;{}_1\succ\; h.$$

**Proof.** Let $T_*^0$ be as in Theorem 7.2. Thus we have $h' \;{}_1\succ\; h \Rightarrow T_*^0(h') > T_*^0(h)$. On the other hand if $T_*^0(h') > T_*^0(h)$ we must have $h' \;{}_1\succ\; h$: By supposition $h'$ and $h$ are ${}_1\succ$-comparable; if $h \;{}_1\succ\; h'$, Theorem 7.1 would require that $T_*^0(h) \geqq T_*^0(h')$.

**Remark 8.2.** In fact, the requirement that $\Sigma$ be a metric space may be replaced by the weaker requirement that $H$ be metrizable (or, equivalently, that $H$ have a countable base of open sets). Recall the hypothesis of Theorem 7.2.

The stipulation that $\Sigma$ be a metric space may be dropped from the hypothesis of Theorem 8.1 altogether provided that $H$ is totally ordered by ${}_3\succ$. In this case we get an even stronger result—that *every* Clausius temperature scale on $H$ preserves the order precisely:

**Theorem 8.2.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system with hotness levels $H$. If $H$ is totally ordered by ${}_3\succ$ (and, in particular, by ${}_4\succ$) then every Clausius temperature scale on $H$ encodes the order precisely; that is, for every $T_* \in \mathscr{T}_*$ we have*

$$T_*(h') > T_*(h) \Leftrightarrow h' \;{}_3\succ h.$$

**Proof.** For any $T_* \in \mathscr{T}_*$ we have by Theorem 7.3 that $h' \;{}_3\succ h \Rightarrow T_*(h') > T_*(h)$. On the other hand if, for some $T_* \in \mathscr{T}_*$, we have $T_*(h') > T_*(h)$ we must also have $h' \;{}_3\succ h$: By supposition $h'$ and $h$ are ${}_3\succ$-comparable; if $h \;{}_3\succ h'$ Theorem 7.3 would require that $T_*(h) > T_*(h')$.

**Remark 8.3.** As we shall see in Appendix D, Theorem 8.2 holds true with ${}_3\succ$ replaced by ${}_2\succ$.

Before stating our next theorem we should recall the background against which it is set. For us the hotness levels of a Kelvin-Planck system $(\Sigma, \mathscr{C})$ were defined objects. They emerged as equivalence classes of states induced in $\Sigma$ solely by the supply $\mathscr{C}$ of cyclic heating measures. The set $H$ of all hotness levels so constructed derived its topology from that of $\Sigma$, but we placed no restriction on the topology of $\Sigma$ other than that it be compact and Hausdorff. Nevertheless, our next theorem asserts that if $\mathscr{C}$ is sufficiently rich in cyclic heating measures as to render any pair of hotness levels comparable with respect to a suitable *hotter than* relation, then *it can only be the case that $H$ is topologically identical to a subset of the real line.*

**Theorem 8.3.** *Let $H$ be the set of hotness levels of a Kelvin-Planck system $(\Sigma, \mathscr{C})$. If either*

(i) *$H$ is totally ordered by ${}_1\succ$ and $\Sigma$ is a metric space*

*or*

(ii) *$H$ is totally ordered by ${}_3\succ$,*

*then $H$ is homeomorphic to a subset of the real line. In particular, if $\Sigma$ is connected then $H$ is homeomorphic to a bounded closed interval of the real line.*

**Proof.** If either (i) or (ii) is satisfied then Theorems 8.1 and 8.2 ensure the existence of a Clausius temperature scale $T_*: H \to \mathrm{P} \subset \mathbb{R}$ such that $T_*(h') \neq T_*(h)$ whenever $h' \neq h$. Thus, the continuous function $T_*$ maps $H$ bijectively onto $T_*(H)$. Moreover, $H$ is compact (Lemma 6.2) and $T_*(H)$, being a subset of the Hausdorff space $\mathbb{R}$, is itself Hausdorff. But any continuous bijective map from a compact space onto a Hausdorff space is a homeomorphism ([B], p. 87). Thus, $T_*$ provides a homeomorphism between $H$ and $T_*(H) \subset \mathbb{R}$.

If $\Sigma$ is connected then $H$ is connected, for $H$ is the image of $\Sigma$ under the continuous function $\pi: \Sigma \to H$. In this case $T_*(H)$, being the image of the compact connected set $H$ under the continuous function $T_*$, must itself be a compact connected subset of the real line, and any such set is a bounded closed interval ([B], p. 337).

**Remark 8.4.** It will follow from results in Appendix D that condition (ii) in Theorem 8.3 can be replaced by the weaker requirement that $H$ be totally ordered by ${}_2\succ$.

**Remark 8.5.** Let $H$ be the set of hotness levels for a Kelvin-Planck system $(\Sigma, \mathscr{C})$. By the ${}_1\succ$*-topology* on $H$ we mean the coarsest topology that renders the sets

$$\{h' \in H \mid h' \;{}_1\!\succcurlyeq h\} \text{ and } \{h' \in H \mid h \;{}_1\!\succcurlyeq h'\}$$

closed for every $h \in H$. Equivalently, the ${}_1\succ$-topology is the coarsest topology on $H$ that renders the sets

$$\{h' \in H \mid h' \;{}_1\!\not\succcurlyeq h\} \text{ and } \{h' \in H \mid h \;{}_1\!\not\succcurlyeq h'\} \tag{8.1}$$

open for every $h \in H$. For the purposes of this remark we call the topology placed on $H$ in Definition 6.2 the *given topology*.

Note that the given topology depends on $\mathscr{C}$ only to the extent that $\mathscr{C}$ induces the equivalence relation ($\sim$) on $\Sigma$ which generates the set $H$ of equivalence classes; thereafter, the given topology is inherited from the topology of $\Sigma$ in the usual manner. On the other hand, the set $\mathscr{C}$ influences the ${}_1\succ$-topology on $H$ in a very direct way, for the partial order ${}_1\succ$ itself derives from $\mathscr{C}$ through Definition 7.1.

At first glance, then, it may seem somewhat surprising that the given topology need bear any relation to the ${}_1\succ$-topology. From results accumulated above, however, it is not difficult to see, at least when $\Sigma$ is a metric space and $H$ is totally ordered by ${}_1\succ$, that the given topology and the ${}_1\succ$-topology are identical. In fact, a similar result holds true whether or not $\Sigma$ is metrizable and whether or not ${}_1\succ$ provides a total order. This we show in the following proposition.

**Proposition 8.3.** *Let $H$ be the set of hotness levels for a Kelvin-Planck system. Then the given topology for $H$ contains the ${}_1\succ$-topology. Moreover, the two topologies are identical if and only if the ${}_1\succ$-topology is Hausdorff.*

**Proof.** As always, we let $\mathscr{T}_*$ denote the set of Clausius temperature scales on $H$. From Theorem 7.1 and Proposition 8.1 it follows that, for any $h \in H$,

$$\{h' \in H \mid h' \;{}_1\!\not\succcurlyeq h\} = \bigcup_{T_* \in \mathscr{T}_*} T_*^{-1}(\{x \in \mathbb{R} \mid x < T_*(h)\})$$

and

$$\{h' \in H \mid h \;{}_1\!\not\succcurlyeq h'\} = \bigcup_{T_* \in \mathscr{T}_*} T_*^{-1}(\{x \in \mathbb{R} \mid x > T_*(h)\}).$$

Since each $T_* \in \mathscr{T}_*$ is continuous with respect to the given topology on $H$ (Lemma 6.3), each of the sets in (8.1) is the union of sets which are open in the given topology. Thus, the sub-base for the ${}_1\succ$-topology described by (8.1) consists of sets which are open in the given topology, and from this it follows that every set which is open in the ${}_1\succ$-topology is open in the given topology.

Since the given topology is Hausdorff (Lemma 6.3), the ${}_1\succ$-topology can coincide with the given topology only if the ${}_1\succ$-topology is Hausdorff. Moreover, if the ${}_1\succ$-topology is Hausdorff then the two topologies coincide: Since $H$ is compact in the given topology (Lemma 6.3) and the ${}_1\succ$-topology is contained in the given topology, $H$ is clearly compact in the ${}_1\succ$-topology. Thus, if the ${}_1\succ$-topology is Hausdorff, then $H$ is compact and Hausdorff in both the given topology and the ${}_1\succ$-topology. It is well known that if two comparable topologies on the same set are both compact and Hausdorff then the two topologies are in fact identical ([B], p. 88).

Before closing this section we should examine the counterexample discussed earlier (Section 5) in light of results we have accumulated. If, in that counterexample, $\Sigma = [1, 2] \times [1, 2]$ is presumed endowed with its usual (metric) topology and if the order posited there is understood even in the sense of our weakest *hotter than* relation (${}_1\succ$), Theorem 8.1 tells us that the counterexample, so interpreted, cannot be realized as a Kelvin-Planck system: If it could, Theorem 8.1 would ensure the existence of a real-valued function that preserves the order precisely, but we proved in Section 5 that the counterexample admits no such function. Thus, with $\Sigma = [1, 2] \times [1, 2]$ there can exist no set of measures $\mathscr{C} \subset \mathscr{M}(\Sigma)$ such that $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system and such that $\mathscr{C}$ induces the order posited.

## 9. Essentially Unique Clausius Temperature Scales and the Role of Carnot Elements

If, for a Kelvin-Planck system $(\Sigma, \mathscr{C})$, $T(\cdot)$ is a Clausius temperature scale on $\Sigma$ then so is $\alpha T(\cdot)$, where $\alpha$ is any positive number. It is natural to ask whether there can exist a Clausius temperature scale for $(\Sigma, \mathscr{C})$ which is not a positive constant multiple of some fixed one. The answer clearly resides in the richness of the set $\mathscr{C}$ of cyclic heating measures for the Kelvin-Planck system under study: The larger the collection of cyclic heating measures, the smaller will be the collection of continuous functions on $\Sigma$ that might qualify as Clausius temperature scales.

Classical arguments suggest that if a Kelvin-Planck system is suitably well endowed with what we shall call *Carnot elements* then uniqueness of a Clausius temperature scale (up to a positive constant multiple) is ensured. We shall prove not only this *but also the converse.* Uniqueness of a Clausius temperature scale *requires* that the Kelvin-Planck system at hand be well endowed with Carnot elements.

It should be emphasized that nothing we have done thus far required any mention of Carnot cycles. In particular the *existence* of a Clausius temperature scale is ensured for every Kelvin-Planck system without the need of any further suppositions. On the other hand, Theorem 9.1 will tell us that any argument ensuring the essential *uniqueness* of a Clausius temperature scale for a particular Kelvin-Planck system must necessarily invoke postulates at least equivalent to an assertion that the system is suitably rich in Carnot elements.

**Remark 9.1.** Consider a Kelvin-Planck system $(\Sigma, \mathscr{C})$ for which all Clausius temperature scales on $\Sigma$ are identical up to multiplication by a positive constant. Theorem 7.4 already provides an indication of how large $\mathscr{C}$ must be: If $H$ is the set of hotness levels for the Kelvin-Planck system under consideration and $\mathscr{T}_*$ is the collection of Clausius scales on $H$, it is clear that all elements of $\mathscr{T}_*$ must also be identical up to multiplication by a positive constant. Thus, if $h'$ and $h$ are hotness levels, the ratio

$$\frac{T_*(h')}{T_*(h)}$$

must be independent of any particular choice of $T_* \in \mathscr{T}_*$, and if $h'$ is distinct from $h$ that ratio must be different from unity (Remark 6.2). Consequently, Theorem 7.4 requires that any pair of distinct hotness levels be ${}_4\succ$-comparable, which is to say that *$H$ must be totally ordered by ${}_4\succ$* (and must therefore be both homeomorphic and order-similar to a subset of the real line). For each pair of distinct hotness levels, then, there must exist in $\hat{\mathscr{C}}$ an element of the kind described in Definition 7.3. As we shall see, for each pair of distinct hotness levels there must in fact exist in $\hat{\mathscr{C}}$ a special element of this kind: a Carnot element operating between them.

**Definition 9.1.** *A* **reversible element** *of a Kelvin-Planck system $(\Sigma, \mathscr{C})$ is a measure $q \in \hat{\mathscr{C}}$ such that $-q$ is also a member of $\hat{\mathscr{C}}$. An* **irreversible element** *of $(\Sigma, \mathscr{C})$ is a member of $\hat{\mathscr{C}}$ that is not reversible.*

Keeping in mind that $\hat{\mathscr{C}} = \mathscr{cl}\,[\mathrm{Cone}\,(\mathscr{C})]$, we note that for $q$ to be a reversible element of $(\Sigma, \mathscr{C})$ neither it nor its negative need be an element of $\mathscr{C}$, the collection of heating measures corresponding to cyclic processes. Rather, we merely require $q$ and $-q$ to be approximated by such measures or at least by their positive multiples. On the other hand, $q$ is an irreversible element of $(\Sigma, \mathscr{C})$ if $q$ is approximated by positive multiples of the cyclic heating measures but $-q$ is not; in this case there is a neighborhood of $-q$ containing no heating measure corresponding to a cyclic process nor any positive multiple of one.

**Definition 9.2.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system. A measure $q \in \hat{\mathscr{C}}$ is a* **Carnot element** *of $(\Sigma, \mathscr{C})$ if $q$ is a reversible element of $(\Sigma, \mathscr{C})$ and, in addition, there exist hotness levels $h'$ and $h$ such that $q$ has a representation*

$$q = \mu' - \mu,$$

*where $\mu'$ and $\mu$ are non-zero elements of $\mathscr{M}_+(\Sigma)$ such that* $\operatorname{supp} \mu' \subset h'$ *and* $\operatorname{supp} \mu \subset h$. *Such a Carnot element* **operates between hotness levels $h'$ and $h$.**

In rough terms a Carnot element is a reversible element which, if viewed as a heating measure, would correspond to a process such that heat absorption is experienced only by material points of a single hotness and such that the same is true for heat emission. Here again we do not insist that a Carnot element be a heating measure corresponding to any cyclic process for the system at hand, only that it and its negative be approximated by such measures (or at least by their positive multiples). This we believe to be compatible with the usual view of Carnot cycles: that they are idealizations of cyclic processes which, if not among the true cyclic processes of a particular system, are in some sense approximated by them.

**Theorem 9.1.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system with hotness levels $H$, and let $T$ be a Clausius temperature scale on $\Sigma$. The following are equivalent:*

(i) *Every Clausius temperature scale on $\Sigma$ is a (positive) constant multiple of $T$.*

(ii) *Any $q \in \mathscr{M}(\Sigma)$ that satisfies*

$$\oint_{\Sigma} \frac{dq}{T} = 0$$

*is an element of $\hat{\mathscr{C}}$.*

(iii) *For each pair of hotness levels $h' \in H$ and $h \in H$ there exists a Carnot element of $(\Sigma, \mathscr{C})$ operating between $h'$ and $h$.*

**Proof.** (i) $\Rightarrow$ (ii): Suppose that (i) holds and that $q \in \mathscr{M}(\Sigma)$ satisfies the equation in (ii). If $q$ is not a member of $\hat{\mathscr{C}}$ then Lemma 6.2 ensures the existence of a Clausius scale $T^\circ$ on $\Sigma$ such that

$$\oint_{\Sigma} \frac{dq}{T^\circ} > 0.$$

Clearly, then, $T^\circ$ cannot be a constant multiple of $T$, and (i) is contradicted.

(ii) $\Rightarrow$ (iii): Let $h'$ and $h$ be any hotness levels in $H$. The desired Carnot element may be constructed as follows: Let $\mu'$ and $\mu$ be any non-zero measures of $\mathscr{M}_+(\Sigma)$ such that $\operatorname{supp} \mu' \subset h'$, $\operatorname{supp} \mu \subset h$, and

$$\frac{\mu'(h')}{\mu(h)} = \frac{T_*(h')}{T_*(h)},$$

where $T_*$ is the Clausius scale induced on $H$ by $T$. (For example, we can take $\mu' = T_*(h')\,\delta_{\sigma'}$ and $\mu = T_*(h)\,\delta_\sigma$, where $\sigma'$ and $\sigma$ are any states in $h'$ and $h$, respectively.) Then

$$q = \mu' - \mu$$

is a Carnot element operating between $h'$ and $h$: To see that $q$ lies in $\hat{\mathscr{C}}$ we invoke (ii) and observe that

$$\begin{aligned}\oint_{\Sigma} \frac{dq}{T} &= \int_{h'} \frac{d\mu'}{T} - \int_{h} \frac{d\mu}{T} \\ &= \frac{\mu'(h')}{T_*(h')} - \frac{\mu(h)}{T_*(h)} \\ &= 0.\end{aligned}$$

The proof that $-q$ lies in $\hat{\mathscr{C}}$ is similar.

(iii) $\Rightarrow$ (i): Suppose that (iii) holds but that (i) does not. Let $T^\dagger$ be a Clausius scale on $\Sigma$ that is not a constant multiple of $T$. Thus, if $T_*^\dagger$ and $T_*$ are the Clausius scales induced on $H$ by $T^\dagger$ and $T$, there must exist a pair of hotness levels $h'$ and $h$ such that

$$\frac{T_*^\dagger(h')}{T_*^\dagger(h)} \neq \frac{T_*(h')}{T_*(h)}. \tag{9.1}$$

Now let $q = \mu' - \mu$ be a Carnot element operating between $h'$ and $h$, where $\mu'$ and $\mu$ are as in Definition 9.2. Since both $q$ and $-q$ are elements of $\hat{\mathscr{C}}$ we must have, by virtue of Remark 4.2, that

$$\begin{aligned} 0 &= \int_\Sigma \frac{dq}{T} \\ &= \int_{h'} \frac{d\mu'}{T} - \int_h \frac{d\mu}{T} \\ &= \frac{\mu'(h')}{T_*(h')} - \frac{\mu(h)}{T_*(h)}. \end{aligned}$$

Hence, we have

$$\frac{T_*(h')}{T_*(h)} = \frac{\mu'(h')}{\mu(h)}. \tag{9.2}$$

Since $T^\dagger$ is also a Clausius scale this same reasoning may be applied to obtain

$$\frac{T_*^\dagger(h')}{T_*^\dagger(h)} = \frac{\mu'(h')}{\mu(h)}. \tag{9.3}$$

But (9.2) and (9.3) taken together contradict (9.1).

**Remark 9.2.** Consider a Kelvin-Planck system $(\Sigma, \mathscr{C})$ for which all Clausius temperature scales are constant multiples of some fixed one, $T$. For any pair of hotness levels $h'$ and $h$, Theorem 9.1 ensures that there will exist a Carnot element operating between $h'$ and $h$. In fact, our proof that (ii) implies (iii) gives something more, that there will generally exist many such Carnot elements: If $\mu'$ and $\mu$ are any non-zero measures of $\mathscr{M}_+(\Sigma)$ such that $\operatorname{supp}\mu' \subset h'$, $\operatorname{supp}\mu \subset h$, and such that (9.2) is satisfied, then $\mu' - \mu$ is a Carnot element operating between $h'$ and $h$. In particular, if $\sigma'$ and $\sigma$ are states in hotness levels $h'$ and $h$, respectively, then $T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma$ must be a Carnot element of $(\Sigma, \mathscr{C})$. That is, *both* $T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma$ *and its negative must be members of* $\hat{\mathscr{C}}$. Since every state resides in some hotness level, it follows that a Kelvin-Planck system $(\Sigma, \mathscr{C})$ admits an essentially *unique* Clausius scale *only if for every state in* $\Sigma$ *there exists a Carnot element in which that state manifests itself.*

**Remark 9.3.** Even for a Kelvin-Planck system not sufficiently rich in Carnot elements to ensure an essentially unique Clausius scale, the *collection* of Clausius scales nevertheless provides information on those Carnot elements that do exist.

Let $h'$ and $h$ be hotness levels and $\mathscr{T}_*$ the collection of Clausius scales on $H$, the set of all hotness levels. *The following statements are equivalent*:

(i) *The ratio*

$$\frac{T_*(h')}{T_*(h)}$$

*is the same for all* $T_* \in \mathscr{T}_*$.

(ii) *There exists a Carnot element operating between* $h'$ *and* $h$.

Proof of this equivalence is similar to that in Theorem 9.1.

Let $(\Sigma, \mathscr{C})$ be an arbitrary Kelvin-Planck system, not necessarily one for which all Clausius scales are identical up to multiplication by a constant. If $T$ is a particular Clausius scale and $q$ is a reversible element of $(\Sigma, \mathscr{C})$ it is an immediate consequence of Remark 4.2 that

$$\int_{\Sigma} \frac{dq}{T} = 0. \tag{9.4}$$

On the other hand, if $q$ is some element of $\mathscr{M}(\Sigma)$ such that (9.4) holds we cannot infer that $q$ is a reversible element of $(\Sigma, \mathscr{C})$, for neither $q$ nor $-q$ need be an element of $\hat{\mathscr{C}}$. If, however, all Clausius scales for $(\Sigma, \mathscr{C})$ are constant multiples of $T$, *then* $q$ *must in fact be a reversible element of* $(\Sigma, \mathscr{C})$. This is the content of our first corollary to Theorem 9.1.

**Corollary 9.1.** *Let* $(\Sigma, \mathscr{C})$ *be a Kelvin-Planck system for which all Clausius temperature scales on* $\Sigma$ *are constant multiples of some fixed one,* $T$. *The reversible elements of* $(\Sigma, \mathscr{C})$ *are precisely those elements of* $\mathscr{M}(\Sigma)$ *that satisfy the equation*

$$\int_{\Sigma} \frac{dq}{T} = 0.$$

**Proof.** It remains only to show that any $q \in \mathscr{M}(\Sigma)$ that satisfies (9.4) must be a reversible element of $(\Sigma, \mathscr{C})$. If $q$ satisfies (9.4) then so must $-q$. The implication (i) $\Rightarrow$ (ii) of Theorem 9.1 then ensures that both $q$ and $-q$ are elements of $\hat{\mathscr{C}}$, whereupon $q$ is a reversible element of $(\Sigma, \mathscr{C})$.

**Remark 9.4.** Even for a Kelvin-Planck system $(\Sigma, \mathscr{C})$ for which the Clausius scales are not constant multiples of some fixed one, the *collection* $\mathscr{T}$ of Clausius scales on $\Sigma$ nevertheless serves to determine all the reversible elements of $(\Sigma, \mathscr{C})$: *The reversible elements of* $(\Sigma, \mathscr{C})$ *are precisely those elements of* $\mathscr{M}(\Sigma)$ *that satisfy the equation*

$$\int_{\Sigma} \frac{dq}{T} = 0 \tag{9.4}$$

*for every* $T \in \mathscr{T}$.

**Proof.** That a reversible element of $(\Sigma, \mathscr{C})$ satisfies the equation above for every $T \in \mathscr{T}$ follows from Remark 4.2. Next suppose that $q \in \mathscr{M}(\Sigma)$ satisfies (9.4) for every $T \in \mathscr{T}$. That $q$ is an element of $\hat{\mathscr{C}}$ follows from Lemma 6.2, for otherwise there would exist $T^\circ \in \mathscr{T}$ such that

$$\int_\Sigma \frac{dq}{T^\circ} > 0,$$

in contradiction to what has been supposed. Moreover, $-q$ must also satisfy (9.4) for every $T \in \mathscr{T}$, and it too must lie in $\hat{\mathscr{C}}$ by similar reasoning. Thus, $q$ is a reversible element of $(\Sigma, \mathscr{C})$.

Corollary 9.1 asserts that a Kelvin-Planck system with an essentially unique Clausius temperature scale must be as rich in *reversible* elements as the Clausius inequality will allow. We may still ask how well supplied with *irreversible* elements such a system might be. Our next corollary asserts that *Kelvin-Planck systems which give rise to essentially unique Clausius scales are of two distinct kinds: those that admit no irreversible elements whatsoever and those that are as rich in irreversible elements as the Clausius inequality will allow.*

**Corollary 9.2.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system for which all Clausius temperature scales are constant multiples of some fixed one, $T$. Of those elements $q \in \mathscr{M}(\Sigma)$ such that*

$$\int_\Sigma \frac{dq}{T} < 0 \tag{9.5}$$

*either all are contained in $\hat{\mathscr{C}}$ or none are.*

**Proof.** We shall prove that if $\hat{\mathscr{C}}$ contains even one element of $\mathscr{M}(\Sigma)$ that satisfies (9.5) it must contain all such elements. Let $q^\dagger$ and $q$ be elements of $\mathscr{M}(\Sigma)$ that satisfy (9.5). We suppose that $q^\dagger$ is a member of $\hat{\mathscr{C}}$ and show that $q$ must be a member of $\hat{\mathscr{C}}$ as well. We begin by writing

$$q = \alpha q^\dagger + (q - \alpha q^\dagger), \tag{9.6}$$

where $\alpha$ is the positive number defined by

$$\alpha = \int_\Sigma \frac{dq}{T} \Big/ \int_\Sigma \frac{dq^\dagger}{T}.$$

Since $\hat{\mathscr{C}}$ is a cone and $q^\dagger$ is a member of $\hat{\mathscr{C}}$ the first term on the right side of (9.6) is also a member of $\hat{\mathscr{C}}$. Moreover, it is easy to verify that

$$\int_\Sigma \frac{1}{T} d(q - \alpha q^\dagger) = 0,$$

whereupon Theorem 9.1, (i) $\Rightarrow$ (ii), ensures that $q - \alpha q^{\dagger}$ is a member of $\hat{\mathscr{C}}$ as well. Thus, (9.6) asserts that $q$ is the sum of two members of $\hat{\mathscr{C}}$. Since $\hat{\mathscr{C}}$ is a convex cone, $q$ must also be a member of $\hat{\mathscr{C}}$.

**Remark 9.5.** If $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system with an essentially unique Clausius temperature scale, then Corollary 9.2 and the implication (i) $\Rightarrow$ (ii) of Theorem 9.1 require that $\hat{\mathscr{C}}$ be either a closed hyperplane or a closed half-space.

Theories of thermodynamics that *begin* with Carnot cycles usually start out by arguing the existence of a temperature scale such that equation (9.4) holds for all Carnot cyles. To show that (9.4) holds for all reversible cyclic processes, whether or not these be Carnot cycles, it is subsequently argued that any reversible cycle can, in some sense, be well approximated by combinations of Carnot cycles. (See, for example, [D2], p. 35.) Underlying such an argument, of course, is the presumption that there exists a supply of Carnot cycles sufficently rich as to approximate whatever reversible cycle might present itself. Although we have had no need to mount an argument of this kind in proving Corollary 9.1, we nevertheless think it interesting that ideas upon which the approximation procedure rest can be rendered precise within the context of the theory presented here. This is the subject of our next corollary.

**Corollary 9.3.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system for which all Clausius scales are constant multiples of some fixed one. The set of all linear combinations of Carnot elements of $(\Sigma, \mathscr{C})$ is dense in the set of all reversible elements of $(\Sigma, \mathscr{C})$.*

**Proof.** Let $T$ be a Clausius temperature scale on $\Sigma$ for the Kelvin-Planck system under consideration, and let $\mathscr{S} \subset \mathscr{M}(\Sigma)$ denote the span of its Carnot elements. When $q^{\dagger}$ is a reversible element of $(\Sigma, \mathscr{C})$, and therefore satisfies

$$\int_{\Sigma} \frac{dq^{\dagger}}{T} = 0, \tag{9.7}$$

we must show that $q^{\dagger}$ is contained in $cl(\mathscr{S})$. Suppose it is not. Then the compact set $\{q^{\dagger}\}$ is disjoint from the closed linear subspace $cl(\mathscr{S})$. In this case Theorem 2.1 ensures the existence of a continuous linear functional on $\mathscr{M}(\Sigma)$ that takes a positive value on $\{q^{\dagger}\}$ and takes the value zero everywhere on $cl(\mathscr{S})$. (Recall the discussion following Theorem 2.1.) Every continuous linear functional on $\mathscr{M}(\Sigma)$ is of the kind $\tilde{\phi}$ for some $\phi \in C(\Sigma, \mathbb{R})$. (Recall the discussion of $\mathscr{M}(\Sigma)$ in Section 2.) Thus, we are ensured the existence of a function $\phi \in C(\Sigma, \mathbb{R})$ such that

$$\int_{\Sigma} \phi \, dq^{\dagger} > 0 \tag{9.8}$$

and

$$\int_{\Sigma} \phi \, dq = 0, \quad \forall\, q \in cl(\mathscr{S}). \tag{9.9}$$

Let $\sigma' \in \Sigma$ be some fixed state and let $\sigma \in \Sigma$ be any other state. It follows from Remark 9.2 that $T(\sigma')\,\delta_{\sigma'} - T(\sigma)\delta_\sigma$ is a Carnot element of $(\Sigma, \mathscr{C})$ and is therefore contained in $\mathscr{S}$. From (9.9) we have

$$\int_\Sigma \phi \, d[T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma] = T(\sigma')\,\phi(\sigma') - T(\sigma)\,\phi(\sigma). \\ = 0.$$

Thus,

$$\phi(\sigma) = \frac{T(\sigma')\,\phi(\sigma')}{T(\sigma)}, \quad \forall\, \sigma \in \Sigma,$$

whereupon $\phi(\cdot)$ is a constant multiple of $1/T(\cdot)$. This, (9.7) and (9.8) taken together provide a contradiction.

## 10. Concluding Remarks

**Remark 10.1.** In Section 3 we suggested that, throughout most of this article, it would be useful to keep in mind a cyclic heating system that derives from consideration of cyclic processes suffered by bodies composed of a prescribed material. In such a case elements of $\Sigma$ would be identified with elements of a *constitutive domain* appropriate to the material at hand.

However, we also alluded to the idea that a cyclic heating system might characterize a theory of cyclic processes suffered by bodies of arbitrary constitution. In this case, means employed to describe the states of material points would be required to transcend features of particular materials. For example, a cyclic heating system might characterize a theory that presupposes the existence of an "empirical temperature scale" which, for *any* substance, associates with each material point a numerical description of its instantaneous "state of hotness". In such a context, $\Sigma$ might be identified with a bounded closed real interval $E$ of empirical temperatures and $\mathscr{C}$ might be identified with heating measures on $E$ corresponding to cyclic processes (suffered by *any* body) wherein no material point experiences an empirical temperature outside $E$. In a similar but more fundamental vein, SERRIN ([S1]–[S3]) presupposes the existence of a *universal hotness manifold* (an oriented continuous one-dimensional real manifold), elements of which are *hotness levels*, and he presumes that for *any* body at any instant each material point has associated with it a hotness level. In this case $\Sigma$ might be identified with a compact set $H$ of hotness levels and $\mathscr{C}$ with heating measures on $H$ corresponding to cyclic processes (suffered by *any* body) wherein no material point experiences a hotness level outside $H$.

With $H$ and $\mathscr{C}$ so interpreted, it is our purpose here to state results of some of our work in a context similar to SERRIN'S. However, *we shall not presume in advance that $H$ has any "one-dimensional" structure*; rather, we shall presume only that $H$ is a compact Hausdorff space. For the rest of this remark we ensure compatibility with the Second Law by requiring that $(H, \mathscr{C})$ be a Kelvin-Planck system. Moreover, we shall also suppose that $(H, \mathscr{C})$ has the following property: For $h' \in H$ and $h \in H$,

$$\delta_{h'} - \delta_h \in \hat{\mathscr{C}} \text{ and } \delta_h - \delta_{h'} \in \hat{\mathscr{C}} \Rightarrow h' = h. \tag{10.1}$$

Property (10.1) ensures that elements of $H$ are in fact hotness levels in the sense of Definitions 6.1 and 6.2. Thus, with (10.1) in force, there is no real distinction between the "space of states" and the "space of hotness levels" for the Kelvin-Planck system $(H, \mathscr{C})$. (In particular, there is no real distinction between the set $\mathscr{T}$ of Clausius scales on the "space of states" and the set $\mathscr{T}_*$ of Clausius scales on the "space of hotness levels.") Note that condition (10.1) is equivalent to the requirement that each pair of distinct hotness levels be distinguished by at least one Clausius temperature scale (Theorem 6.1).

In the following statements we summarize consequences for the Kelvin-Planck system $(H, \mathscr{C})$ of some, but not all, of the theorems contained in Sections 4–9.

A. From Theorem 4.1: *There exists for $(H, \mathscr{C})$ a Clausius temperature scale. That is, there exists a continuous function $T: H \to \mathbb{P}$ such that*

$$\int_H \frac{dq}{T} \leqq 0, \quad \forall\, q \in \mathscr{C}.$$

Note that this result derives directly from the Kelvin-Planck property by way of the Hahn-Banach Theorem; no appeal is made to the existence of special materials or of special processes.

B. From Theorems 8.2 and 8.3: *If $H$ is totally ordered by ${}_3\!\succ$ then it can only be the case that $H$ is both homeomorphic and order-similar to a subset of the real line; in particular, if $H$ is connected then $H$ is both homeomorphic and order-similar to a bounded closed interval of the real line. In fact, if $T(\cdot)$ is any Clausius scale for $(H, \mathscr{C})$ then $T(\cdot)$ provides a homeomorphism between $H$ and $T(H)$. Moreover, $T(\cdot)$ preserves order faithfully; that is,*

$$h' \,{}_3\!\succ h \Leftrightarrow T(h') > T(h).$$

C. From Theorem 9.1 and its corollaries: *Let $T: H \to \mathbb{P}$ be a Clausius temperature scale for $(H, \mathscr{C})$. Then all Clausius scales for $(H, \mathscr{C})$ are constant multiples of $T(\cdot)$ if and only if both $T(h')\,\delta_{h'} - T(h)\,\delta_h$ and its negative are members of $\hat{\mathscr{C}}$ for every pair $h', h \in H$. If $T(\cdot)$ is an essentially unique Clausius temperature scale for $(H, \mathscr{C})$ then the reversible elements of $(H, \mathscr{C})$ are precisely those $q \in \mathscr{M}(H)$ that satisfy*

$$\int_H \frac{dq}{T} = 0; \tag{10.2}$$

*in particular, if $q \in \mathscr{M}(H)$ satisfies* (10.2) *then both $q$ and $-q$ are members of $\hat{\mathscr{C}}$. Moreover, if $T(\cdot)$ is an essentially unique Clausius scale and $(H, \mathscr{C})$ admits at least one irreversible element then the irreversible elements of $(H, \mathscr{C})$ are precisely those $q \in \mathscr{M}(H)$ that satisfy*

$$\int_H \frac{dq}{T} < 0; \tag{10.3}$$

*in particular, if* $q \in \mathscr{M}(H)$ *satisfies* (10.3) *it must be the case that* $q$ *is a member of* $\hat{\mathscr{C}}$.

**Remark 10.2.** Our emphasis throughout has been on existence and properties of *continuous* functions that serve as Clausius temperature scales for a cyclic heating system $(\Sigma, \mathscr{C})$. When $\Sigma$ has enough structure so that it makes sense to speak about differentiation of functions defined on $\Sigma$, it may be important to know about existence and properties of Clausius scales which are not only continuous but also possess a specified degree of smoothness. In fact, we can restrict attention to Clausius scales having features of this kind by modifying the topology on $\mathscr{M}(\Sigma)$ in an appropriate way.

We state the relevant facts abstractly. Suppose that $\mathscr{F}$ is a linear subspace of $C(\Sigma, \mathbb{R})$, not necessarily closed, such that for any pair of distinct elements in $\mathscr{M}(\Sigma)$, say $\mu$ and $\nu$, there exists $\phi \in \mathscr{F}$ for which

$$\int_\Sigma \phi \, d\mu \neq \int_\Sigma \phi \, d\nu .$$

(Note that when $\Sigma$ is a differentiable manifold, $\mathscr{F}$ may be taken to be the subspace of real-valued functions on $\Sigma$ that are $k$-times continuously differentiable, for such functions separate measures as indicated.) The *$\mathscr{F}$-topology* on $\mathscr{M}(\Sigma)$ is the coarsest topology which, for every $\phi \in \mathscr{F}$, renders continuous the linear functional $\tilde{\phi}$ on $\mathscr{M}(\Sigma)$ defined by

$$\tilde{\phi}(\mu) \equiv \int_\Sigma \phi \, d\mu .$$

Clearly the $\mathscr{F}$-topology is contained in the weak-star topology (Section 2) and, in fact, coincides with the weak-star topology when $\mathscr{F}$ is taken to be $C(\Sigma, \mathbb{R})$. Since the $\mathscr{F}$-topology is no finer than the weak-star topology, every weak-star compact set in $\mathscr{M}(\Sigma)$ is also $\mathscr{F}$-compact. Moreover, when $\mathscr{M}(\Sigma)$ is given the $\mathscr{F}$-topology every continuous linear functional on $\mathscr{M}(\Sigma)$ is of the form $\tilde{\phi}$ for some $\phi \in \mathscr{F}$. Thus $\mathscr{M}(\Sigma)$, endowed with the $\mathscr{F}$-topology, has the essential features used in proofs of theorems in preceding sections, except that elements of $\mathscr{F}$ assume the role formerly played by elements of $C(\Sigma, \mathbb{R})$.

The topology of $\mathscr{M}(\Sigma)$ enters our analysis of a cyclic heating system $(\Sigma, \mathscr{C})$ primarily through the definition of $\hat{\mathscr{C}}$. Recall that $\hat{\mathscr{C}}$ is the closure of Cone $(\mathscr{C})$. Had we given $\mathscr{M}(\Sigma)$ the $\mathscr{F}$-topology and interpreted the closure operation accordingly, $\hat{\mathscr{C}}$ would be no smaller and could in fact be larger than it was when $\mathscr{M}(\Sigma)$ carried the weak-star topology. Thus, with $\hat{\mathscr{C}}$ interpreted in the sense of the $\mathscr{F}$-topology, the Kelvin-Planck condition (Definition 3.2) is either identical to or stronger than its weak-star version.

For the rest of this remark we presume that $\mathscr{M}(\Sigma)$ has the $\mathscr{F}$-topology, and we interpret $\hat{\mathscr{C}}$ and the Kelvin-Planck condition in the corresponding sense. In this case the proof of Theorem 4.1 is easily modified to show that *a cyclic heating system* $(\Sigma, \mathscr{C})$ *is a Kelvin-Planck system if and only if there exists* $T: \Sigma \to \mathbb{P}$

*such that* $1/T$ *is a member of* $\mathscr{F}$ *and*

$$\int_{\Sigma} \frac{dq}{T} \leqq 0, \quad \forall q \in \mathscr{C}.$$

(Note that if $\Sigma$ is a differentiable manifold and $\mathscr{F}$ is taken to be the subspace of all real-valued functions on $\Sigma$ that admit $k$ continuous derivatives, then $1/T$ lies in $\mathscr{F}$ only if $T$ itself lies in $\mathscr{F}$.) Provided that we strengthen our definition of a Clausius temperature scale to include the requirement that $1/T$ be a member of $\mathscr{F}$, certain important theorems in preceding sections hold true as written. In particular, this is the case for Theorems 6.1 and 9.1, but it should be kept in mind that $\hat{\mathscr{C}}$ requires the $\mathscr{F}$-topology interpretation in Definitions 6.1, 9.1, and 9.2.

**Remark 10.3.** Although our concern here has been with questions pertaining to the Clausius inequality, we intend to address issues connected with the more general Clausius-Duhem inequality in a separate study. To a great extent SERRIN'S casting of the Clausius inequality in terms of heating measures made possible the analysis contained in this article, and we wish to suggest here how the Clausius-Duhem inequality might be similarly cast in order that it might be studied in a similar way. For this purpose we believe there is considerable advantage in joining SERRIN'S use of heating measures to describe heat receipt by a body during the course of a process with means employed by NOLL [N2] to describe the condition of the body as the process progresses.

Consider a thermodynamical theory of bodies composed of a prescribed material, and suppose that states of material points are identified with elements of a Hausdorff space. By way of analogy to work in preceding sections we restrict our attention to processes (not necessarily cyclic) wherein no material point experiences a state outside some compact set $\Sigma$. With each such process we associate a *heating measure*, $q \in \mathscr{M}(\Sigma)$, interpreted just as in Section 3. Moreover, at any instant during the process we can identify the *condition* of the body suffering the process with a measure $m \in \mathscr{M}_+(\Sigma)$ to be interpreted in the following way: For each Borel set $B \subset \Sigma$, the number $m(B)$ is the mass of that part of the body consisting of material points experiencing states contained within $B$. In particular, $m(\Sigma)$ is the mass of the entire body. With $m_i$ denoting the initial condition of the body (at the beginning of the process) and $m_f$ denoting its final condition (at the end of the process), the measure $m_f - m_i \in \mathscr{M}(\Sigma)$ describes the *change of condition* the body experiences upon suffering the process in question. To reflect the idea that the body should admit no change in its total mass we require that $m_f - m_i$ be a member of the linear subspace

$$\mathscr{M}^\circ(\Sigma) := \{\mu \in \mathscr{M}(\Sigma) \mid \mu(\Sigma) = 0\}.$$

Thus, with each process admitted by the theory under consideration we can associate a pair

$$(m_f - m_i, q) \in \mathscr{M}^\circ(\Sigma) \oplus \mathscr{M}(\Sigma),$$

where $m_f - m_i$ is the change of condition in the body suffering the process and $q$ is the heating measure for the process. For the purposes of this remark, then, we can identify all processes admitted by the theory with a set

$$\mathsf{P} \subset \mathcal{M}^\circ(\Sigma) \oplus \mathcal{M}(\Sigma).$$

In essence, the Clausius-Duhem inequality amounts to an assertion that there exist (continuous) functions $T: \Sigma \to \mathbb{P}$ and $s: \Sigma \to \mathbb{R}$ such that

$$\int_\Sigma s \, d(m_f - m_i) - \int_\Sigma \frac{dq}{T} \geqq 0, \quad \forall \, (m_f - m_i, q) \in \mathsf{P}. \tag{10.4}$$

The function $T(\cdot)$ is a *Clausius-Duhem temperature scale* and $s(\cdot)$ is a *specific entropy function*. That is, for each $\sigma \in \Sigma$, $T(\sigma)$ and $s(\sigma)$ give the temperature and entropy per mass at a material point in state $\sigma$. If $m_i$ and $m_f$ give the initial and final condition of a body suffering a particular process, then the *initial entropy* of the body is

$$\int_\Sigma s \, dm_i,$$

and its *final entropy* is

$$\int_\Sigma s \, dm_f;$$

thus, the first integral on the left of (10.4) is the *change of entropy* the process induces in the body.

We believe that issues connected with the Clausius-Duhem inequality can be profitably addressed in terms of the following question: *How is the nature of the set* P *related to the existence and properties of pairs* $(s(\cdot), T(\cdot))$ *that satisfy* (10.4)? It is a question of this kind that we intend to take up in the future.*

## Appendix A. The Convexity of $\hat{C}$

Here we prove Proposition 3.1. We wish to show that if $\mathcal{P}_c \subset \mathbb{P} \times \mathcal{M}(\Sigma)$ enjoys Properties 1–3 listed in Section 3 and if the set $\mathscr{C}$ is as defined just before the statement of Proposition 3.1, then the set

$$\hat{\mathscr{C}} := cl\,(\text{Cone}\,(\mathscr{C}))$$

is convex. Because $\hat{\mathscr{C}}$ is a cone it is enough to show that if $q_1$ and $q_2$ lie in $\hat{\mathscr{C}}$ then so does $q_1 + q_2$.

Ideas underlying the proof might be described in rough terms as follows: If $q_1$ and $q_2$ are cyclic heating measures (i.e., members of $\mathscr{C}$) corresponding to two processes of identical duration then Property 1 gives the desired result immediately. If the durations $\tau_1$ and $\tau_2$ are not identical but $\tau_1/\tau_2$ is a rational number then Properties 1 and 2 can be used in conjunction to show that $q_1 + q_2$ is a member of Cone ($\mathscr{C}$). If the durations are not rationally related then Property 3

* Work by COLEMAN, OWEN, and SERRIN on thermodynamic foundations underlying the Clausius-Duhem inequality has appeared very recently. See [COS] and [O]. For a summary of results aling the lines sketched above see our appendix in [T3].

can be invoked to assert the existence of a cyclic heating measure $\varphi_1' \in \mathscr{C}$ which is close to $\varphi_1$ and which corresponds to a process of duration $\tau_1'$, where $\tau_1'$ is close to $\tau_1$ and is rationally related to $\tau_2$. Thus, $\varphi_1 + \varphi_2$ is close to $\varphi_1' + \varphi_2$, which in turn is a member of Cone ($\mathscr{C}$). In this way we can show that $\varphi_1 + \varphi_2$ is a member of $cl$ (Cone ($\mathscr{C}$)). When $\varphi_1$ and $\varphi_2$ are not both members of $\mathscr{C}$ but are members of $\hat{\mathscr{C}}$ the fact that $\varphi_1 + \varphi_2$ lies in $\hat{\mathscr{C}}$ follows from the fact that $\varphi_1$ and $\varphi_2$ are approximated by members of Cone ($\mathscr{C}$).

**Proof of Proposition 3.1.** In our proof we use the following notation: With $k$ a positive number, $\nu$ an element of $\mathscr{M}(\Sigma)$, and $\Omega$ a subset of $\mathscr{M}(\Sigma)$ we take the sets $k\Omega \subset \mathscr{M}(\Sigma)$ and $\nu + \Omega \subset \mathscr{M}(\Sigma)$ to be defined by

$$k\Omega := \{k\mu \in \mathscr{M}(\Sigma) \mid \mu \in \Omega\},$$

$$\nu + \Omega := \{\nu + \mu \in \mathscr{M}(\Sigma) \mid \mu \in \Omega\}.$$

We proceed by way of two lemmas.

**Lemma A.1.** *If $\varphi_1$ and $\varphi_2$ are members of $\mathscr{C}$ and $n_1$ and $n_2$ are positive integers, then $n_1\varphi_1 + n_2\varphi_2$ is a member of $\hat{\mathscr{C}}$. In particular, $\varphi_1 + \varphi_2$ is a member of $\hat{\mathscr{C}}$.*

**Proof.** Since $\varphi_1$ and $\varphi_2$ are members of $\mathscr{C}$ we have that $(\tau_1, \varphi_1) \in \mathscr{P}_c$ and $(\tau_2, \varphi_2) \in \mathscr{P}_c$ for positive numbers $\tau_1$ and $\tau_2$. Let $\Omega$ be any neighborhood of zero in $\mathscr{M}(\Sigma)$. By Property 3 there is a $\delta > 0$ such that for each $\tau_1'$ with $|\tau_1' - \tau_1| < \delta$, there exists

$$\varphi_1' \in \varphi_1 + \left(\frac{1}{n_1}\right)\Omega$$

such that $(\tau_1', \varphi_1') \in \mathscr{P}_c$. Now let $m_1$ and $m_2$ be positive integers such that

$$\left|\frac{m_2}{m_1} - \frac{\tau_1}{\tau_2}\right| < \frac{\delta}{\tau_2}.$$

Thus,

$$\left|\frac{m_2\tau_2}{m_1} - \tau_1\right| < \delta,$$

whereupon for some $\varphi_1' \in \varphi_1 + (1/n_1)\,\Omega$ we have $(m_2\tau_2/m_1, \varphi_1') \in \mathscr{P}_c$. By Property 2 we therefore have

$$(m_2\tau_2, m_1\varphi_1') \in \mathscr{P}_c,$$

and, again by Property 2, we also have

$$(m_2\tau_2, m_2\varphi_2) \in \mathscr{P}_c.$$

Now we invoke Property 1 to assert that

$$(m_2\tau_2, m_1m_2n_1\varphi_1') \text{ and } (m_2\tau_2, m_1m_2n_2\varphi_2)$$

are also members of $\mathscr{P}_c$. From this and Property 1 we deduce that

$$(m_2\tau_2, m_1m_2[n_1\varphi_1' + n_2\varphi_2])$$

is a member of $\mathscr{P}_c$ as well. Thus, $m_1 m_2[n_1 \varphi_1' + n_2 \varphi_2]$ is a member of $\mathscr{C}$, from which it follows that

$$n_1 \varphi_1' + n_2 \varphi_2 \in \text{Cone}(\mathscr{C}).$$

Since $\varphi_1' \in \varphi_1 + (1/n_1)\,\Omega$ we therefore have that the set

$$(n_1 \varphi_1 + n_2 \varphi_2) + \Omega$$

has a non-empty intersection with Cone $(\mathscr{C})$. We conclude that

$$n_1 \varphi_1 + n_2 \varphi_2 \in cl\,(\text{Cone}(\mathscr{C})).$$

**Lemma A.2.** *If $\varphi_1$ and $\varphi_2$ are members of* Cone $(\mathscr{C})$ *then $\varphi_1 + \varphi_2$ is a member of $\hat{\mathscr{C}}$.*

**Proof.** We first show that if $\varphi_1'$ and $\varphi_2'$ are members of $\mathscr{C}$ and $\beta_1$ and $\beta_2$ are positive rational numbers then $\beta_1 \varphi_1' + \beta_2 \varphi_2'$ is a member of $\hat{\mathscr{C}}$. Let $\beta_1 = n_1/m_1$ and $\beta_2 = n_2/m_2$, where $n_1, n_2, m_1$, and $m_2$ are positive integers. Then

$$\beta_1 \varphi_1' + \beta_2 \varphi_2' = \frac{1}{m_1 m_2}[n_1 m_2 \varphi_1' + n_2 m_1 \varphi_2'].$$

Because $\varphi_1'$ and $\varphi_2'$ are members of $\mathscr{C}$ it follows from Lemma A.1 that the element in brackets is a member of $\hat{\mathscr{C}}$. Since $\hat{\mathscr{C}}$ is a cone, the equation above implies that $\beta_1 \varphi_1' + \beta_2 \varphi_2'$ is an element of $\hat{\mathscr{C}}$.

Now suppose that $\varphi_1$ and $\varphi_2$ are members of Cone $(\mathscr{C})$. Then there exist positive numbers, $\alpha_1$ and $\alpha_2$, and elements of $\mathscr{C}$, $\varphi_1'$ and $\varphi_2'$, such that $\varphi_1 = \alpha_1 \varphi_1'$ and $\varphi_2 = \alpha_2 \varphi_2'$. Thus

$$\varphi_1 + \varphi_2 = \alpha_1 \varphi_1' + \alpha_2 \varphi_2'.$$

If $\alpha_1$ and $\alpha_2$ are both rational we have from the argument above that $\varphi_1 + \varphi_2 \in \hat{\mathscr{C}}$. If $\alpha_1$ and $\alpha_2$ are not both rational then any neighborhood of $\alpha_1 \varphi_1' + \alpha_2 \varphi_2'$ contains an element $\beta_1 \varphi_1' + \beta_2 \varphi_2'$, where $\beta_1$ and $\beta_2$ are rational. But we have already argued that such an element lies in $\hat{\mathscr{C}}$. Thus, every neighborhood of $\varphi_1 + \varphi_2$ meets $\hat{\mathscr{C}}$. Since $\hat{\mathscr{C}}$ is closed $\varphi_1 + \varphi_2$ must lie in $\hat{\mathscr{C}}$. This completes the proof of Lemma A.2.

To complete the proof of Proposition 3.1 we suppose that $\varphi_1$ and $\varphi_2$ are members of $\hat{\mathscr{C}}$ and show that $\varphi_1 + \varphi_2$ is also a member of $\hat{\mathscr{C}} = cl\,(\text{Cone}(\mathscr{C}))$. Because $\mathscr{M}(\Sigma)$ is locally convex it suffices to show that if $\Omega$ is any convex neighborhood of zero then $(\varphi_1 + \varphi_2) + \Omega$ meets the closed set $\hat{\mathscr{C}}$. Let $\Omega$ be such a neighborhood. Since $\varphi_1$ and $\varphi_2$ lie in $cl\,(\text{Cone}(\mathscr{C}))$ the sets $\varphi_1 + (1/2)\,\Omega$ and $\varphi_2 + (1/2)\,\Omega$ must meet Cone $(\mathscr{C})$. Thus, there exist elements $\nu_1, \nu_2 \in \Omega$ and $\varphi_1', \varphi_2' \in$ Cone $(\mathscr{C})$ such that

$$\varphi_1 + (1/2)\,\nu_1 = \varphi_1' \text{ and } \varphi_2 + (1/2)\,\nu_2 = \varphi_2'.$$

Therefore

$$(\varphi_1 + \varphi_2) + \left(\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2\right) = \varphi_1' + \varphi_2'.$$

By Lemma A.2, $\varphi_1' + \varphi_2'$ is a member of $\hat{\mathscr{C}}$. Since $\Omega$ is convex, $(1/2)\,\nu_1 + (1/2)\,\nu_2$ is an element of $\Omega$. Thus, the set $(\varphi_1 + \varphi_2) + \Omega$ meets $\hat{\mathscr{C}}$, whereupon $\varphi_1 + \varphi_2$ is a member of $\hat{\mathscr{C}}$. This completes the proof of Proposition 3.1.

**Remark A.1.** Compactness of $\Sigma$ plays no essential role in the proof of Proposition 3.1. It is included in the statement of the proposition only to reflect the mathematical setting established in Sections 2 and 3.

## Appendix B. Existence of a ${}_1\succ$-Monotonic Clausius Temperature Scale

**Proof of Theorem 7.2.** We need a few facts from topology: A compact Hausdorff space has a countable base of open sets if and only if it is metrizable ([B, Part 2], Prop. 16, p. 158). A metrizable space (not necessarily compact) has a countable base of open sets if and only if it contains a countable dense set ([B, Part 2], Prop. 14, p. 156). If $X$ is a compact metrizable space then $C(X, \mathbb{R})$, endowed with the metric topology induced by the norm $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$, contains a countable dense set {[D4], p. 140).

By virtue of Lemma 6.3 (b), $H$ is a compact Hausdorff space. If $H$ has a countable base of open sets (or, equivalently, is metrizable) then $C(H, \mathbb{R})$ contains a countable dense set. Thus, the metric space $C(H, \mathbb{R})$ has a countable base of open sets. This gives a countable base for the relative (metric) topology on $\mathscr{T}_* \subset C(H, \mathbb{R})$. Hence $\mathscr{T}_*$ contains a countable dense set, say $\{T_*^{(1)}, T_*^{(2)}, \ldots\}$.

Now, for $n = 1, 2, \ldots,$ let the function $\phi_n : H \to \mathbb{P}$ be defined by

$$\phi_n(h) \equiv 1/T_*^{(n)}(h).$$

Moreover, let $\phi_0 : H \to \mathbb{P}$ be defined by

$$\phi_0 = \sum_{n=1}^{\infty} \frac{\phi_n}{2^n \|\phi_n\|_\infty}.$$

The series converges uniformly so that $\phi_0$ is continuous. Since, for $n = 1, 2, \ldots,$ we have

$$\int_\Sigma \phi_n \circ \pi \, d\varphi \leqq 0, \quad \forall\, \varphi \in \hat{\mathscr{C}},$$

we also have

$$\int_\Sigma \phi_0 \circ \pi \, d\varphi \leqq 0, \quad \forall\, \varphi \in \hat{\mathscr{C}}.$$

Therefore the function

$$T_*^0 := 1/\phi_0$$

is a Clausius scale on $H$.

Next we show that $T_*^\circ$ has the desired property. Let $h', h \in H$ be such that $h' \,{}_1\!\succ h$. We wish to show that $T_*^\circ(h') > T_*^\circ(h)$. Theorem 7.1 ensures the existence of a $T_* \in \mathscr{T}_*$ such that $T_*(h') > T_*(h)$. Since $\{T_*^{(1)}, T_*^{(2)}, \ldots\}$ is dense in $\mathscr{T}_*$

there is a $T_*^{(n)}$ close enough to $T_*$ so that $T_*^{(n)}(h') > T_*^{(n)}(h)$ and $\phi_n(h') < \phi_n(h)$. Moreover, Theorem 7.1 ensures that $\phi_k(h') \leqq \phi_k(h)$, $k = 1, 2, \ldots$. It follows that $\phi_0(h') < \phi_0(h)$ and $T_*^\circ(h') > T_*^\circ(h)$.

To prove the last assertion of Theorem 7.2 we observe that if the compact set $\Sigma$ is a metric space then it has a countable base of open sets. So then does $H$ (Lemma 6.3e).

## Appendix C. Addendum on the Relation ${}_3\succ$

Our purpose in this appendix is two-fold. First we show in Example C.1 that if $h'$ and $h$ are hotness levels of a Kelvin-Planck system such that $T_*(h') > T_*(h)$ for all $T_* \in \mathscr{T}_*$ it need not be the case that $h'$ and $h$ are ${}_3\succ$-comparable. This same example also serves to demonstrate that the implication (i) $\Rightarrow$ (ii) of Proposition 8.2 cannot be reversed (Remark C.1). Second, we show in Proposition C.1 that ${}_3\succ$ gives a partial order to the hotness levels of a Kelvin-Planck system.

**Example C.1.** Let $\Sigma$ be a set consisting of three elements $\sigma_1, \sigma_2,$ and $\sigma_3$. We give $\Sigma$ the discrete topology; thus, $\Sigma$ is compact and Hausdorff. Moreover, $C(\Sigma, \mathbb{R})$ is identical to the set of all real-valued functions on $\Sigma$, and we give the three-dimensional vector space $C(\Sigma, \mathbb{R})$ its usual (norm) topology.

Let $\mathscr{F} \subset C(\Sigma, \mathbb{R})$ be defined by

$$\mathscr{F} := \{f \in C(\Sigma, \mathbb{R}) \mid f \geqq 0,\ f(\sigma_2) \geqq \sqrt{3} f(\sigma_3),\ \ f(\sigma_1) \geqq [(f(\sigma_2))^2 + (f(\sigma_3))^2]^{1/2}\}.$$

It is not difficult to see that $\mathscr{F}$ is a closed convex cone.

Also let $\mathscr{C} \subset \mathscr{M}(\Sigma)$ be defined by

$$\mathscr{C} := \left\{ q \in \mathscr{M}(\Sigma) \,\middle|\, \int_\Sigma f\, dq \leqq 0, \quad \forall f \in \mathscr{F} \right\}.$$

Note that $\mathscr{C}$ is a closed convex cone so that $\mathscr{C}$ is identical to the set $\hat{\mathscr{C}} := cl$ (Cone $(\mathscr{C})$).

Because $\mathscr{F}$ contains (strictly) positive elements it follows easily that $\hat{\mathscr{C}}$ contains no non-zero positive measure. Thus $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system. Moreover, we show in Appendix G that

$$\int_\Sigma f\, dq \leqq 0,\ \forall\, q \in \hat{\mathscr{C}} \ \Rightarrow\ f \in \mathscr{F}.$$

Therefore the set $\mathscr{T}$ of all Clausius temperature scales on $\Sigma$ coincides with the set of functions obtained by taking reciprocals of strictly positive functions in $\mathscr{F}$. Since the positive functions of $\mathscr{F}$ separate points of $\Sigma$ it follows from Theorem 6.1 that no two points in $\Sigma$ lie in the same hotness level. Thus, the hotness levels may be identified with points of $\Sigma$, and there is no real distinction between Clausius scales defined on $\Sigma$ and Clausius scales defined on the set $H$ of hotness levels.

Now let $f$ be a strictly positive element of $\mathscr{F}$. From the definition of $\mathscr{F}$ it follows that $f(\sigma_1) > f(\sigma_2) > f(\sigma_3)$. Thus, for the Kelvin-Planck system $(\Sigma, \mathscr{C})$

we have

$$T(\sigma_3) > T(\sigma_2) > T(\sigma_1), \qquad \forall\, T \in \mathscr{T}. \tag{C.1}$$

We wish to show that $\sigma_2$ and $\sigma_1$ are not ${}_3\succ$-comparable. If $\sigma_2$ and $\sigma_1$ are ${}_3\succ$-comparable we must have, by Theorem 7.3, that $\sigma_2 \,{}_3\!\succ \sigma_1$, which requires the existence in $\hat{\mathscr{C}}$ of an element $\bar{\varphi} = \delta_{\sigma_2} - \delta_{\sigma_1} + \nu$ with $\nu \in \mathscr{M}_+(\Sigma) \setminus \{0\}$. The set $\mathscr{F}$ clearly contains an element $f_0$ such that $f_0(\sigma_1) = f_0(\sigma_2) > 0$ and $f_0(\sigma_3) = 0$. If $\bar{\varphi}$ is a member of $\hat{\mathscr{C}}$ we must have

$$0 \geqq \int_\Sigma f_0\, d\bar{\varphi} = f_0(\sigma_2) - f_0(\sigma_1) + \int_\Sigma f_0\, d\nu = \int_\Sigma f_0\, d\nu.$$

This implies that $\operatorname{supp} \nu = \{\sigma_3\}$. Thus, for any $f \in \mathscr{F}$ we have

$$0 \geqq \int_\Sigma f\, d\bar{\varphi} = f(\sigma_2) - f(\sigma_1) + \nu(\{\sigma_3\}) f(\sigma_3). \tag{C.2}$$

Now for any $0 \leqq \Theta \leqq \pi/6$ the function $f \in C(\Sigma, \mathbb{R})$ defined by $f(\sigma_1) = 1$, $f(\sigma_2) = \cos \Theta$ and $f(\sigma_3) = \sin \Theta$ is a member of $\mathscr{F}$. For such an element (C.2) takes the form

$$0 \geqq \cos \Theta - 1 + \nu(\{\sigma_3\}) \sin \Theta. \tag{C.3}$$

For any positive value of $\nu(\{\sigma_3\})$, however, (C.3) fails to obtain for sufficiently small positive values of $\Theta$ (since $\cos \Theta - 1 \approx \Theta^2/2$, $\sin \Theta \approx \Theta$). Hence we have a contradiction.

**Remark C.1.** Example C.1 serves to demonstrate that the implication (i) $\Rightarrow$ (ii) of Proposition 8.2 cannot be reversed. By virtue of the inequalities (C.1) the Kelvin-Planck system constructed in Example C.1 has the property that, for every Clausius scale, the isotherms induced in $\Sigma$ are identical to hotness levels (in this case the sets $\{\sigma_1\}$, $\{\sigma_2\}$ and $\{\sigma_3\}$). Nevertheless, the hotness levels are not totally ordered by ${}_3\succ$: in particular, $\{\sigma_1\}$ and $\{\sigma_2\}$ are not ${}_3\succ$-comparable.

**Proposition C.1.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system with hotness levels $H$. The relation ${}_3\succ$ is a partial order on $H$.*

**Proof.** Antisymmetry has already been proved in Remark 7.7. We show here that ${}_3\succ$ is transitive. Suppose that $h'' \,{}_3\!\succ h'$ and $h' \,{}_3\!\succ h$. From Definition 7.2 it follows that there exist measures

$$\mu'' - \mu_1' + \nu_1 \in \hat{\mathscr{C}} \tag{C.4}$$

and

$$\mu_2' - \mu + \nu_2 \in \hat{\mathscr{C}}, \tag{C.5}$$

where $\mu''$, $\mu_1'$, $\nu_1$, $\mu_2'$, $\mu$, and $\nu_2$ are all-non zero elements of $\mathscr{M}_+(\Sigma)$ with

$$\operatorname{supp} \mu'' \subset h'', \operatorname{supp} \mu_1' \subset h', \mu''(h'') = \mu_1'(h')$$

and

$$\operatorname{supp} \mu_2' \subset h', \operatorname{supp} \mu \subset h, \mu_2'(h') = \mu(h).$$

Moreover, since $\hat{\mathscr{C}}$ is a cone there is no loss of generality in taking

$$\mu_1'(h') = \mu_2'(h'). \tag{C.6}$$

Because $\hat{\mathscr{C}}$ is a convex cone we can add (C.4) and (C.5) to obtain

$$\mu'' - \mu + (\nu_1 + \nu_2) + (\mu_2' - \mu_1') \in \hat{\mathscr{C}}. \tag{C.7}$$

To show that $h'' \,_3\!\succ h$ it suffices to show that $\mu_1' - \mu_2'$ is a member of $\hat{\mathscr{C}}$, for then this measure can be added to (C.7) to produce an element in $\hat{\mathscr{C}}$ of the kind required by Definition 7.2. That $\mu_1' - \mu_2'$ is an element of $\hat{\mathscr{C}}$ is an easy consequence of Lemma 6.2 since, for every $T \in \mathscr{T}$, we have

$$\int_\Sigma \frac{1}{T} d[\mu_1' - \mu_2'] = \frac{1}{T_*(h')}[\mu_1'(h') - \mu_2'(h')] = 0.$$

## Appendix D. The Relation $_2\!\succ$

Recall that if $h'$ and $h$ are hotness levels of a Kelvin-Planck system $(\Sigma, \mathscr{C})$, then $h'$ is hotter than $h$ in the first sense $(h' \,_1\!\succ h)$ if there exists

$$\varphi = \mu' - \mu + \nu \in \hat{\mathscr{C}}, \tag{D.1}$$

where $\mu$ and $\mu'$ are elements of $\mathscr{M}_+(\Sigma)$ such that

$$\operatorname{supp} \mu' \subset h', \operatorname{supp} \mu \subset h, \mu'(h') = \mu(h) > 0, \tag{D.2}$$

and

$$\nu \in \mathscr{M}_+(\Sigma). \tag{D.3}$$

Recall also that if there exists $\varphi \in \hat{\mathscr{C}}$ satisfying all of the above with

$$\nu \in \mathscr{M}_+(\Sigma) \setminus \{0\} \tag{D.4}$$

then $h'$ is hotter than $h$ in the third sense $(h' \,_3\!\succ h)$.

Although there does not seem to be much room between the relations $_1\!\succ$ and $_3\!\succ$, the condition

$$T_*(h') > T_*(h), \quad \forall\, T_* \in \mathscr{T}_* \tag{D.5}$$

is somewhat stronger than the condition $h' \,_1\!\succ h$ (recall Theorem 7.1) but yet is not sufficiently strong as to ensure that $h' \,_3\!\succ h$. (In particular, we have seen in Appendix C that (D.5) does not imply that $h'$ and $h$ are $_3\!\succ$-comparable.) Here we posit yet another *hotter than* relation ($_2\!\succ$) which is intermediate in strength between $_1\!\succ$ and $_3\!\succ$ and which, like the others, is defined solely in terms of the supply $\mathscr{C}$ of cyclic heating measures. In fact we shall prove that the condition $h' \,_2\!\succ h$ is *equivalent* to (D.5).

To begin we let $h'$ and $h$ be hotness levels of a Kelvin-Planck system $(\Sigma, \mathscr{C})$, and we define

$$\mathscr{R}(h', h) := \{\mu' - \mu \in \mathscr{M}(\Sigma) \mid \mu' \text{ and } \mu \text{ satisfy (D.2)}\}.$$

It is not difficult to see that $\mathscr{R}(h', h)$ is a convex cone. Moreover, by expressing (D.1) in the form $\nu = \varphi - (\mu' - \mu)$, we also see that $h' \,{}_3\!\succ h$ if and only if

$$[\mathscr{M}_+(\Sigma) \setminus \{0\}] \cap [\hat{\mathscr{C}} - \mathscr{R}(h', h)] \neq \emptyset. \tag{D.6}$$

The following definition weakens (D.6) slightly by enlarging the set $[\hat{\mathscr{C}} - \mathscr{R}(h', h)]$.

**Definition D.1.** *For a Kelvin-Planck system* $(\Sigma, \mathscr{C})$ *with hotness levels* $H$ *we say that* $h' \in H$ **is hotter than** $h \in H$ **in the second sense** (*denoted* $h' \,{}_2\!\succ h$) *if* $h' \neq h$ *and*

$$[\mathscr{M}_+(\Sigma) \setminus \{0\}] \cap cl\, [\hat{\mathscr{C}} - \mathscr{R}(h', h)] \neq \emptyset.$$

Thus, we have $h' \,{}_2\!\succ h$ if, for some fixed non-zero $\nu \in \mathscr{M}_+(\Sigma)$, there exists in $\hat{\mathscr{C}}$ an element which is approximated by measures of the form $\mu' - \mu + \nu$, where $\mu'$ and $\mu$ satisfy (D.2).

It is easy to see that

$$h' \,{}_3\!\succ h \Rightarrow h' \,{}_2\!\succ h.$$

The implication

$$h' \,{}_2\!\succ h \Rightarrow h' \,{}_1\!\succ h$$

will result from Theorem 7.1 and the following theorem:

**Theorem D.1.** *Let* $h'$ *and* $h$ *be hotness levels for the Kelvin-Planck system* $(\Sigma, \mathscr{C})$. *Then the following are equivalent:*

(i) $h' \,{}_2\!\succ h$.

(ii) $T_*(h') > T_*(h)$ *for every Clausius temperature scale* $T_* \in \mathscr{T}_*$.

**Proof.** In proving Theorem D.1 we shall draw upon a proposition which will be used not only here but also in Appendix E. For this reason we state and prove the proposition separately in Appendix F. At this point the reader might wish to see Appendix F or at least glance at Corollary F.1.

We prove the equivalence of (i) and (ii) in Theorem D.1 by showing that their negations are equivalent. The negations take the form

(i)′ $[\mathscr{M}_+(\Sigma) \setminus \{0\}] \cap cl\, [\hat{\mathscr{C}} - \mathscr{R}(h', h)] = \emptyset$.

(ii)′ *There exists* $T_* \in \mathscr{T}_*$ *such that* $T_*(h') \leqq T_*(h)$.

Statement (ii)′, however, is equivalent to

(iii)′ *There exists* $T \in \mathscr{T}$ *such that*

$$\int_\Sigma \frac{1}{T}\, d(\mu' - \mu) \geqq 0, \quad \forall\, \mu' - \mu \in \mathscr{R}(h', h).$$

The equivalence of (ii)′ and (iii)′ follows from the fact that if $\mu' - \mu$ is an element of $\mathscr{R}(h, h)$ and $T$ is a Clausius scale on $\Sigma$, then

$$\int_{\Sigma} \frac{1}{T}\, d(\mu' - \mu) = \frac{\mu'(h')}{T_*(h')} - \frac{\mu(h)}{T_*(h)} = \mu(h)\left[\frac{1}{T_*(h')} - \frac{1}{T^*(h)}\right].$$

The equivalence of (i)′ and (iii)′ is an immediate consequence of Corollary F.1. (Identify $\mathscr{B}$ in Corollary F.1 with $\mathscr{R}(h', h)$.) Thus (i)′ and (ii)′ are both equivalent to (iii)′.

**Corollary D.1.** *Let $h'$ and $h$ be hotness levels for a Kelvin-Planck system $(\Sigma, \mathscr{C})$. Then*

$$h' \,{}_2\!\succ h \Rightarrow h' \,{}_1\!\succ h.$$

**Proof.** If $h' \,{}_2\!\succ h$ Theorem D.1 requires that $T_*(h') > T_*(h)$ for every $T_* \in \mathscr{T}_*$. This implies condition (ii) of Theorem 7.1, which in turn implies that $h' \,{}_1\!\succ h$.

**Corollary D.2.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system with hotness levels $H$. Then ${}_2\!\succ$ is a partial order on $H$.*

**Proof.** Both antisymmetry and transitivity are easy consequences of condition (ii) of Theorem D.1.

The following corollary improves Proposition 8.2.

**Corollary D.3.** *Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system, let $H$ be its set of hotness levels, and let $\mathscr{T}$ be its collection of Clausius temperature scales on $\Sigma$. Then the following are equivalent:*

(i) *$H$ is totally ordered by ${}_2\!\succ$.*

(ii) *For each $T \in \mathscr{T}$ every $T$-isotherm coincides with a hotness level. That is, $i_T(\sigma) = \pi(\sigma)$ for every $T \in \mathscr{T}$ and every $\sigma \in \Sigma$.*

**Proof.** We remind the reader that results of Section 6 ensure that, for any Clausius scale $T$, each hotness level resides entirely within a $T$-isotherm. Thus, (ii) amounts to an assertion that, for any $T \in \mathscr{T}$, each $T$-isotherm resides entirely within a hotness level.

Suppose that (i) holds but (ii) does not. Then there exist states $\sigma'$ and $\sigma$ belonging to distinct hotness levels $h'$ and $h$ such that $T(\sigma') = T(\sigma)$ for some $T \in \mathscr{T}$. With $T_*$ the Clausius scale induced on $H$ by $T$ we have $T_*(h') = T_*(h)$. Because $h'$ and $h$ are distinct and ${}_2\!\succ$-comparable Theorem D.1 requires that $T_*(h') \neq T_*(h)$. Thus we have a contradiction.

Next suppose that (ii) holds but (i) does not. Let $h'$ and $h$ be distinct hotness levels which are not ${}_2\!\succ$-comparable. Then Theorem D.1 ensures the existence of Clausius scales $T_*^1, T_*^2 \in \mathscr{T}_*$ such that

$$T_*^1(h') \geqq T_*^1(h) \text{ and } T_*^2(h') \leqq T_*^2(h).$$

In the case that inequality holds in both of the above we can, as in the proof of Proposition 8.1, construct $T_*^3 \in \mathscr{T}_*$ such that $T_*^3(h') = T_*^3(h)$. In any case, then, there exists $T_* \in \mathscr{T}_*$ such that $T_*(h') = T_*(h)$. Thus, the distinct hotness levels $h'$ and $h$ reside in the same isotherm of the Clausius scale $T_* \circ \pi \in \mathscr{T}$. This contradicts (ii).

**Remark D.1.** Even when $H$ is not totally ordered by ${}_2\succ$ we may still assert the equivalence of the following:

(i) *Hotness level $h$ is ${}_2\succ$-comparable to all other hotness levels.*

(ii) *For each $T \in \mathscr{T}$ hotness level $h$ coincides with a $T$-isotherm.*

Proof of this equivalence is similar to the proof of Corollary D.3.

**Corollary D.4.** *Let $H$ be the set of hotness levels of a Kelvin-Planck system $(\Sigma, \mathscr{C})$. If $H$ is totally ordered by ${}_2\succ$ then $H$ is homeomorphic and order-similar to a subset of the real line. Moreover, every Clausius temperature scale $T_* \in \mathscr{T}_*$ reflects the order precisely and provides a homeomorphism between $H$ and $T_*(H)$. In particular, if $\Sigma$ is connected then $H$ is homeomorphic and order-similar to a closed and bounded interval of the real line.*

**Proof.** The proof is virtually identical to that of similar results obtained in Section 8 under the stronger hypothesis that $H$ is totally ordered by ${}_3\succ$. There the essential ingredient was the implication (i) $\Rightarrow$ (ii) of Theorem 7.3. The implication (i) $\Rightarrow$ (ii) of Theorem D.1 gives the required improvement.

## Appendix E. Non-Compact State Spaces

In the main body of this article the state space $\Sigma$ was taken to be compact. Here we study consequences of relaxing this constraint. Counterexamples will be presented to demonstrate that, without modification, neither Theorem 4.1 (existence of Clausius scales) nor Theorem 9.1 (uniqueness of Clausius scales) obtain for non-compact $\Sigma$. We shall, however, prove generalizations of these theorems which are appropriate for situations in which $\Sigma$ is locally compact. Although no attempt will be made to reconstruct all results of Sections 4–9 for locally compact $\Sigma$, we believe that what we do here should provide a reasonably clear indication of how such a reconstruction can be effected.

*Throughout this appendix $\Sigma$ will always be a locally compact Hausdorff space.* In Section 2 we established some vocabulary for the case in which $\Sigma$ is compact, and we need to indicate how that vocabulary should be amended to accommodate the more general situation in which $\Sigma$ is locally compact. In particular, we require some care in specifying what we mean by the vector space $\mathscr{M}(\Sigma)$.

In Section 2 we defined the support of a positive measure on $\Sigma$. Here we extend that definition to embrace non-positive measures as well. Let $\mu$ be a regular real Borel measure on $\Sigma$, and let $\Omega \subset \Sigma$ be an open set. We say that $\mu$ *is zero on* $\Omega$ if, for every Borel set $B \subset \Omega$, we have $\mu(B) = 0$. By the *support* of $\mu$ we mean the subset of $\Sigma$ defined by

$$\operatorname{supp} \mu := \Sigma \setminus (\cup \{\Omega \subset \Sigma \mid \Omega \text{ is open and } \mu \text{ is zero on } \Omega\}).$$

It is clear that $\operatorname{supp} \mu$ is closed; it may or may not be compact.

By $\mathscr{M}(\Sigma)$ we mean here the vector space of regular real (signed) Borel measures on $\Sigma$ with compact support. Note that when $\Sigma$ is compact the definition of $\mathscr{M}(\Sigma)$ given here coincides with that given in Section 2. By $\mathscr{M}_+(\Sigma)$ we mean as before the convex cone consisting of positive measures in $\mathscr{M}(\Sigma)$ (including the zero measure), and by $\mathscr{M}_+^1(\Sigma)$ we mean the convex set in $\mathscr{M}_+(\Sigma)$ consisting of measures of mass one. For compact $K \subset \Sigma$ we define

$$\begin{aligned} \mathscr{M}^K(\Sigma) &:= \{\mu \in \mathscr{M}(\Sigma) \mid \operatorname{supp} \mu \subset K\}, \\ \mathscr{M}_+^K(\Sigma) &:= \{\mu \in \mathscr{M}_+(\Sigma) \mid \operatorname{supp} \mu \subset K\}, \end{aligned} \tag{E.1}$$

and

$$\mathscr{M}_+^{K,1}(\Sigma) := \{\mu \in \mathscr{M}_+^1(\Sigma) \mid \operatorname{supp} \mu \subset K\}.$$

These sets are easily shown to be convex.

Recall that for each $\phi \in C(\Sigma, \mathbb{R})$ the function $\tilde{\phi} : \mathscr{M}(\Sigma) \to \mathbb{R}$ is defined by

$$\tilde{\phi}(\nu) \equiv \int_\Sigma \phi \, d\nu.$$

As in Section 2, we give $\mathscr{M}(\Sigma)$ the coarsest topology that renders $\tilde{\phi}$ continuous for every $\phi \in C(\Sigma, \mathbb{R})$. When $\Sigma$ is locally compact it remains the case that $\mathscr{M}(\Sigma)$ is a locally convex Hausdorff topological vector space, and every continuous real-valued linear function on $\mathscr{M}(\Sigma)$ is of the kind $\tilde{\phi}$ for some $\phi \in C(\Sigma, \mathbb{R})$.

There is, however, an important property $\mathscr{M}(\Sigma)$ has when $\Sigma$ is compact that does not carry over when we require only that $\Sigma$ be locally compact: *With $\Sigma$ locally compact it is not generally the case that $\mathscr{M}_+^1(\Sigma)$ is compact.* Compactness of $\mathscr{M}_+^1(\Sigma)$ entered the proofs of virtually all theorems in Sections 4–9, and it is this lack of compactness that we must confront in this appendix. Nevertheless, when $\Sigma$ is locally compact $\mathscr{M}_+^1(\Sigma)$ is closed and, if $K \subset \Sigma$ is compact, we have that $\mathscr{M}_+^{K,1}(\Sigma)$ is compact.

Here we take a *cyclic heating system* to be as in Definition 3.1 with "locally compact" replacing "compact".* As in Definition 3.2 we take a *Kelvin-Planck system* to be a cyclic heating system for which

$$\hat{\mathscr{C}} \cap \mathscr{M}_+(\Sigma) = \{0\}, \tag{E.2}$$

where

$$\hat{\mathscr{C}} := cl\,[\text{Cone}\,(\mathscr{C})]. \tag{E.3}$$

---

* The interpretation of the sets $\Sigma$ and $\mathscr{C}$ is the same as that given in Section 3. The fact that $\mathscr{C}$ is a subset of $\mathscr{M}(\Sigma)$ implies that all elements of $\mathscr{C}$ have compact support. This we think is natural, given the interpretation of elements of $\mathscr{C}$. Regarding the convexity of $\hat{\mathscr{C}}$ when $\Sigma$ is not compact, the reader might wish to see Remark A.1 of Appendix A.

By a Clausius temperature scale for a cyclic heating system $(\Sigma, \mathscr{C})$ we mean, as before, a continuous function $T: \Sigma \to \mathrm{P}$ such that

$$\int_{\Sigma} \frac{d\mathscr{q}}{T} \leqq 0, \quad \forall \mathscr{q} \in \mathscr{C}. \tag{E.4}$$

The following example demonstrates that, if $\Sigma$ is not compact, a Kelvin-Planck system need not admit a Clausius temperature scale.

**Example E.1.** Let $\Sigma = [0, \infty)$ and, for $x \in \Sigma$, let

$$\phi_1(x) \equiv x, \quad \phi_2(x) \equiv 1 - x^2.$$

Moreover, take $\mathscr{F} \subset C(\Sigma, \mathbb{R})$ and $\mathscr{C} \subset \mathscr{M}(\Sigma)$ to be defined by

$$\mathscr{F} := \{\phi \in C(\Sigma, \mathbb{R}) \mid \phi = s\phi_1 + t\phi_2, s \geqq 0, t \geqq 0\}$$

and

$$\mathscr{C} := \left\{ \mathscr{q} \in \mathscr{M}(\Sigma) \,\middle|\, \int_{\Sigma} \phi \, d\mathscr{q} \leqq 0, \quad \forall \phi \in \mathscr{F} \right\}.$$

We show in Appendix G that $\mathscr{C}$ is a closed convex cone (i.e., $\hat{\mathscr{C}} = \mathscr{C}$) and that

$$\phi \in C(\Sigma, \mathbb{R}) \text{ and } \int_{\Sigma} \phi \, d\mathscr{q} \leqq 0, \ \forall \mathscr{q} \in \hat{\mathscr{C}} \ \Rightarrow \ \phi \in \mathscr{F}. \tag{E.5}$$

To see that $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system (i.e., that (E.2) holds) we suppose that $\mathscr{q}^\circ$ is contained in $\hat{\mathscr{C}} \cap \mathscr{M}_+(\Sigma)$. Then we must have

$$\int_{\Sigma} \phi_1 \, d\mathscr{q}^\circ \leqq 0. \tag{E.6}$$

Since $\mathscr{q}^\circ$ lies in $\mathscr{M}_+(\Sigma)$ and $\phi_1$ is positive on $(0, \infty)$ it follows from (E.6) that

$$\operatorname{supp} \mathscr{q}^\circ \subset \{0\}. \tag{E.7}$$

We must also have that

$$0 \geqq \int_{\Sigma} \phi_2 \, d\mathscr{q}^\circ = \phi_2(0)\, \mathscr{q}^\circ(\{0\}) = \mathscr{q}^\circ(\{0\}). \tag{E.8}$$

Since $\mathscr{q}^\circ$ lies in $\mathscr{M}_+(\Sigma)$, (E.8) can hold only if

$$\mathscr{q}^\circ(\{0\}) = 0. \tag{E.9}$$

But (E.7) and (E.9) ensure that $\mathscr{q}^\circ = 0$ so that $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system.

Now if $T: \Sigma \to \mathrm{P}$ is a Clausius temperature scale for the Kelvin-Planck system $(\Sigma, \mathscr{C})$, (E.5) requires that $1/T$ be a member of $\mathscr{F}$. But $\mathscr{F}$ contains no strictly positive functions. In fact, every element of $\mathscr{F}$ takes the value zero somewhere on $\Sigma$ so that no element of $\mathscr{F}$ admits a reciprocal. Thus, the Kelvin-Planck system $(\Sigma, \mathscr{C})$ admits no Clausius temperature scale.

Example E.1 demonstrates that a Kelvin-Planck system must have additional properties before we can be sure that it admits a Clausius scale. In preparation for a discussion of what such properties might be, it will be useful to review the considerations that led to our definition of a Kelvin-Planck system in the first place (Section 3).

To ensure that a cyclic heating system $(\Sigma, \mathscr{C})$ be compatible with the Kelvin-Planck statement of the Second Law we began by imposing the condition

$$\mathscr{C} \cap \mathscr{M}_+(\Sigma) \text{ is at most the zero measure.} \tag{E.10}$$

This, we argued, is equivalent to the condition

$$\text{Cone}(\mathscr{C}) \cap \mathscr{M}_+(\Sigma) \text{ is at most the zero measure.} \tag{E.11}$$

For the purpose of our discussion here we now wish to assert that (E.10) and (E.11) are equivalent to yet another condition:*

$$[\text{Cone}(\mathscr{C}) - \mathscr{M}_+(\Sigma)] \cap \mathscr{M}_+(\Sigma) \text{ is at most the zero measure.} \tag{E.12}$$

That (E.12) implies (E.11) is an easy consequence of the fact that $\text{Cone}(\mathscr{C})$ lies in $\text{Cone}(\mathscr{C}) - \mathscr{M}_+(\Sigma)$. Moreover (E.11) implies (E.12), for if (E.12) does not hold then there exist $\varphi \in \text{Cone}(\mathscr{C})$, $\nu \in \mathscr{M}_+(\Sigma)$ and a non-zero $\nu' \in \mathscr{M}_+(\Sigma)$ such that $\varphi - \nu = \nu'$; thus $\varphi = \nu + \nu'$, which contradicts (E.11).

In Section 3 we wanted our definition of a Kelvin-Planck system to carry the implication that cyclic processes not only respect the Second Law but also that they do not come arbitrarily close to standing in violation of it. We considered two days in which this requirement might be made precise. First, we strengthened (E.10) to insist that

$$\mathscr{cl}(\mathscr{C}) \cap \mathscr{M}_+(\Sigma) \text{ is at most the zero measure;} \tag{E.13}$$

and, second, we strengthened (E.11) to require that

$$\mathscr{cl}[\text{Cone}(\mathscr{C})] \cap \mathscr{M}_+(\Sigma) = \{0\}. \tag{E.14}$$

Here we also strengthen (E.12) by writing

$$\mathscr{cl}[\text{Cone}(\mathscr{C}) - \mathscr{M}_+(\Sigma)] \cap \mathscr{M}_+(\Sigma) = \{0\}. \tag{E.15}$$

Despite the fact that conditions (E.10)–(E.12) are equivalent, conditions (E.13)–(E.15) are not. In our consideration of Example 3.3 we showed that, when $\mathscr{C}$ is not a cone, (E.14) may be stronger than (E.13). In rough terms, the central idea there was that cyclic heating measures could come arbitrarily close to being "positive in direction" even when the cyclic heating measures themselves do not

* By $\text{Cone}(\mathscr{C}) - \mathscr{M}_+(\Sigma)$ we mean the algebraic difference, not the set theoretic difference. That is, $\mu \in \text{Cone}(\mathscr{C}) - \mathscr{M}_+(\Sigma)$ if and only if $\mu = \varphi - \nu$ with $\varphi \in \text{Cone}(\mathscr{C})$ and $\nu \in \mathscr{M}_+(\Sigma)$. It is useful to keep in mind that $\mathscr{M}_+(\Sigma)$ contains the zero measure; thus, $\text{Cone}(\mathscr{C})$ is contained in $\text{Cone}(\mathscr{C}) - \mathscr{M}_+(\Sigma)$.

come arbitrarily close to being positive. With this in mind, we argued that there is some advantage to posing statements of the Second Law in terms of directions in the vector space $\mathscr{M}(\Sigma)$ along which cyclic heating measures lie. As we shall see, (E.15) may be stronger than (E.14) even when $\mathscr{C}$ is a closed cone.

In order that we might discuss the distinction between conditions (E.14) and (E.15) it will be useful to make precise the notions of "direction," "positive direction," and also the idea that one direction is "less positive" than another. By the *direction* of an element $\mu \in \mathscr{M}(\Sigma)$ we mean the set Cone $(\{\mu\})$—that is, the set of all measures of the form $c\mu$ with $c$ a positive number. Thus, the direction of $\mu$ is the set of all elements of the vector space $\mathscr{M}(\Sigma)$ that "point along" $\mu$. Note that Cone $(\mathscr{C})$ is the union of all directions of cyclic heating measures; these we call the *cyclic heating directions.* The direction of any non-zero element of $\mathscr{M}_+(\Sigma)$ we call a *positive direction.* Note that $\mathscr{M}_+(\Sigma) \setminus \{0\}$ is the union of all positive directions. We shall say that the direction Cone $(\{\mu\})$ is *less positive* than the direction Cone $(\{\mu'\})$ if

$$\text{Cone}\,(\{\mu\}) \subset \text{Cone}\,(\{\mu'\}) - (\mathscr{M}_+(\Sigma) \setminus \{0\}).$$

Thus, a direction is contained in

$$\text{Cone}\,(\mathscr{C}) - [\mathscr{M}_+(\Sigma) \setminus \{0\}]$$

if and only if it less positive than a cyclic heating direction. The union of all such directions taken together with the cyclic heating directions is then

$$\text{Cone}\,(\mathscr{C}) - \mathscr{M}_+(\Sigma).$$

In rough terms, a direction lies in the cone

$$\hat{\mathscr{C}} := cl\,[\text{Cone}\,(\mathscr{C})]$$

if and only if it comes arbitrarily close to being a cyclic heating direction. Similarly, a direction lies in the cone

$$\tilde{\mathscr{C}} := cl\,[\text{Cone}\,(\mathscr{C}) - \mathscr{M}_+(\Sigma)]$$

if and only if it comes arbitrarily close to being a cyclic heating direction or to being less positive than a cyclic heating direction.

In terms of language we now have available, condition (E.11) asserts that no cyclic heating direction is positive. Condition (E.12) asserts not only this but also that no direction less positive than a cyclic heating direction is positive. This amendment is clearly redundant, for (E.11) and (E.12) are equivalent.

In rough terms, (E.14) asserts that cyclic heating directions do not come arbitrarily close to being positive. Condition (E.15) asserts this and also that directions less positive than cyclic heating directions do not come arbitrarily close to being positive. Although the last amendment might appear to be redundant, it is not. This we show in the following remark.

**Remark E.1.** With $\Sigma$ and $\mathscr{C}$ as in Example E.1 we have already demonstrated that $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system, which is to say that condition (E.14) is satisfied. We show here that $(\Sigma, \mathscr{C})$ does not satisfy condition (E.15). Recall that, for Example E.1, $\mathscr{C}$ is a closed cone; in particular, we have $\mathscr{C} = \text{Cone}(\mathscr{C})$. Now for every positive integer $n$ let $\varphi_n \in \mathscr{M}(\Sigma)$ be defined by

$$\varphi_n = \frac{1}{2}\delta_0 - \frac{2}{n}\delta_1 + \frac{1}{n^2}\delta_n. \tag{E.16}$$

By integrating $\varphi_n$ against $\phi_1$ and $\phi_2$ in Example E.1 it is not difficult to verify that

$$\{\varphi_n\}_{n=1,2,\ldots} \subset \mathscr{C}. \tag{E.17}$$

Since $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system, (E.17) implies that

$$cl\,[\{\varphi_n\}_{n=1,2,\ldots}] \text{ contains no non-zero positive measure.} \tag{E.18}$$

Moreover, (E.17) implies that

$$\left\{\varphi_n - \frac{1}{n^2}\delta_n\right\}_{n=1,2,\ldots} \subset \mathscr{C} - \mathscr{M}_+(\Sigma). \tag{E.19}$$

Note that

$$\varphi_n - \frac{1}{n^2}\delta_n = \frac{1}{2}\delta_0 - \frac{2}{n}\delta_1. \tag{E.20}$$

From this it is not difficult to see that every neighborhood of $\frac{1}{2}\delta_0$ contains an element of the set on the left side of (E.19). Thus, we have

$$\frac{1}{2}\delta_0 \in cl\left[\left\{\varphi_n - \frac{1}{n^2}\delta_n\right\}_{n=1,2,\ldots}\right]. \tag{E.21}$$

From (E.19) and (E.21) it follows that

$$\frac{1}{2}\delta_0 \in cl\,[\mathscr{C} - \mathscr{M}_+(\Sigma)] = cl\,[\text{Cone}(\mathscr{C}) - \mathscr{M}_+(\Sigma)].$$

Hence, condition (E.15) is not satisfied by the Kelvin-Planck system $(\Sigma, \mathscr{C})$.

Comparison of (E.18) and (E.21) illustrates a counterintuitive property that the space $\mathscr{M}(\Sigma)$ may have when $\Sigma$ is not compact. The set $\{\varphi_n\}_{n=1,2,\ldots}$ contains no non-zero positive measure, nor does this set approximate any such measure. Yet, by adding the *negative* measure $-n^{-2}\delta_n$ to $\varphi_n$ for each $n = 1, 2, \ldots$, we obtain a set which approximates the *positive* measure $\frac{1}{2}\delta_0$.*

**Remark E.2.** Comments at the end of the preceding remark lay the groundwork for a different, less formal, and somewhat more physical discussion of the condition (E.15). The idea here is that we can, if we wish, view (E.15) not only as a prohibition against certain kinds of behavior in cyclic processes but also as a constraint on the behavior of a more general category of processes. These we shall call *subcyclic processes*.

---

* The difference in behavior of the sequences defined by (E.16) and (E.20) resides in the fact that the sequence $n^{-2}\delta_n$ does not converge to the zero measure: Note that for each $n$ the measure $n^{-2}\delta_n$ integrates the function $f(x) = x^2$ to unity, while the zero measure integrates the same function to zero.

Consider a theory in which states of material points are identified with elements of a locally compact Hausdorff space $\Sigma$, and suppose that to every process admitted by the theory there corresponds a heating measure in $\mathcal{M}(\Sigma)$ with physical interpretation as described in Section 3. The cyclic heating measures $\mathcal{C}$ are, as before, those heating measures which derive from cyclic processes. We say that a process (not necessarily cyclic) is *absorptive* [resp., *emissive*] if the heating measure for the process is a member of $\mathcal{M}_+(\Sigma)$ $[-\mathcal{M}_+(\Sigma)]$; if, in addition, the heating measure is non-zero then the process is *non-trivially absorptive* [*emissive*]. Any process wherein the body suffering the process emits [absorbs] no heat to [from] its environment is absorptive [emissive].

Before defining what we mean by a subcyclic process we give some examples. Consider

(a) a process which consists of a cyclic process followed by an emissive process;

(b) a process which is the union* of a cyclic process and an emissive process;

(c) a process which, when followed by some absorptive process, results in a cyclic process.

It is not difficult to see that heating measures for the processes described are of the form $q - \nu$, where $q$ is a member of $\mathcal{C}$ and $\nu$ is a member of $\mathcal{M}_+(\Sigma)$. By a *subcyclic process* we mean a process with heating measure representable in the form $q - \nu$, where $q$ lies in $\mathcal{C}$ and $\nu$ lies in $\mathcal{M}_+(\Sigma)$. Since the zero measure is contained in $\mathcal{M}_+(\Sigma)$ it follows that every cyclic process is also subcyclic.

Note that if $q'$ is the heating measure for a subcyclic process, then there exists a cyclic process with heating measure $q$ such that $q(B) \geqq q'(B)$ for every Borel set $B \subset \Sigma$; moreover, if the subcyclic process is not cyclic then $q(B) > q'(B)$ for some Borel set $B$. This is to say that if we restrict our attention to heat absorbed by material points experiencing states in some fixed but arbitrary Borel set, then the heat so absorbed during the cyclic process will be no less than that absorbed during the subcyclic process; if the subcyclic process is not cyclic then, for some choice of Borel set, the heat absorbed during the cyclic process will actually be greater. In this sense, every subcyclic process is itself cyclic or else is "less absorptive" than some cyclic process.

Translated into language introduced here, our observations at the end of Remark E.1 indicate that, even though no sequence of heating measures corresponding to cyclic processes converges to a non-zero positive measure, it may still happen that a sequence of heating measures corresponding to subcyclic processes converges to a non-zero positive measure. In somewhat rougher terms this mean that, even when the cyclic processes do not themselves come arbitrarily close to being non-trivially absorptive, there may exist subcyclic processes, each of which is *less* absorptive than some cyclic process, which come arbitrarily close to being non-trivially absorptive! This possibility is not precluded by the Kelvin-Planck condition (E.14) but is precluded by the stronger condition (E.15), for it is easy to see that the heating measure for any subcyclic process lies in Cone $(\mathcal{C})$ $- \mathcal{M}_+(\Sigma)$.

* We use the word *union* in SERRIN's sense. See the discussion of Property 1 in Section 3.

Without going into any detail we assert that, in the presence of reasonable assumptions about the supply of emissive processes (not necessarily cyclic), the set

$$\tilde{\mathscr{C}} := \mathscr{cl}\,[\text{Cone}\,(\mathscr{C}) - \mathscr{M}_+(\Sigma)]$$

will not only contain but will also be identical to the set of measures approximated by heating measures for subcyclic processes. Although we shall not mention subcyclic processes again, readers might wish to keep this interpretation of $\tilde{\mathscr{C}}$ in mind.

**Remark E.3.** In light of Remark E.1 we can see in another way that any attempt to construct a Clausius temperature scale in Example E.1 could not have succeeded: For a cyclic heating system $(\Sigma, \mathscr{C})$ to admit a Clausius temperature scale it is *necessary* that condition (E.15) obtain.

To show this we first show that, if $T: \Sigma \to \mathbb{P}$ is a Clausius scale for $(\Sigma, \mathscr{C})$, then we must have not only (E.4) but also

$$\int_\Sigma \frac{d\mu}{T} \leqq 0, \qquad \forall \mu \in \tilde{\mathscr{C}} := \mathscr{cl}\,[\text{Cone}\,(\mathscr{C}) - \mathscr{M}_+(\Sigma)]. \tag{E.22}$$

Note that (E.4) implies

$$\int_\Sigma \frac{d\varphi}{T} \leqq 0, \qquad \forall \varphi \in \text{Cone}\,(\mathscr{C}). \tag{E.23}$$

Moreover, the positivity of $T$ implies

$$\int_\Sigma \frac{d\nu}{T} \geqq 0, \qquad \forall \nu \in \mathscr{M}_+(\Sigma). \tag{E.24}$$

Combining (E.22) and (E.23), we obtain

$$\int_\Sigma \frac{d\mu}{T} \leqq 0, \qquad \forall \mu \in \text{Cone}\,(\mathscr{C}) - \mathscr{M}_+(\Sigma). \tag{E.25}$$

Since the function

$$\mu \in \mathscr{M}(\Sigma) \mapsto \int_\Sigma \frac{d\mu}{T}$$

is continuous, (E.25) implies (E.22).

From this we conclude that if a cyclic heating system $(\Sigma, \mathscr{C})$ admits a Clausius scale $T(\cdot)$, then $\tilde{\mathscr{C}}$ can contain no nonzero positive measure: Any such measure integrates $1/T(\cdot)$ positively, but (E.22) asserts that all elements of $\tilde{\mathscr{C}}$ integrate $1/T(\cdot)$ non-positively. Thus the existence of a Clausius scale for $(\Sigma, \mathscr{C})$ requires that (E.15) hold.

Motivated by our considerations thus far, we record the following definition:

**Definition E.1.** *A* **strong Kelvin-Planck system** *is a cyclic heating system* $(\Sigma, \mathscr{C})$ *such that*

$$\tilde{\mathscr{C}} \cap \mathscr{M}_+(\Sigma) = \{0\}, \tag{E.26}$$

*where*

$$\tilde{\mathscr{C}} := c\ell\,[\mathrm{Cone}\,(\mathscr{C}) - \mathscr{M}_+(\Sigma)]. \tag{E.27}$$

**Remark E.4.** As we shall see, results obtained for Kelvin-Planck systems with $\Sigma$ compact carry over (sometimes under slightly restricted conditions) to strong Kelvin-Planck systems provided that $\tilde{\mathscr{C}}$ replaces $\hat{\mathscr{C}}$ whenever $\hat{\mathscr{C}}$ appears in the statement of those results. For this reason it will be helpful if we make the relationship between $\tilde{\mathscr{C}}$ and $\hat{\mathscr{C}}$ somewhat more explicit than we have thus far. It is not difficult to show that

$$c\ell\,[\mathrm{Cone}\,(\mathscr{C}) - \mathscr{M}_+(\Sigma)] = c\ell\,[c\ell\,(\mathrm{Cone}\,(\mathscr{C})) - \mathscr{M}_+(\Sigma)]. \tag{E.28}$$

Thus, we have

$$\tilde{\mathscr{C}} = c\ell\,(\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)). \tag{E.29}$$

Since both $\hat{\mathscr{C}}$ and $\mathscr{M}_+(\Sigma)$ are convex cones it follows easily from (E.29) that $\tilde{\mathscr{C}}$ is a closed convex cone. Although both $\hat{\mathscr{C}}$ and $\mathscr{M}_+(\Sigma)$ are closed, it need not be the case that $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is closed. It is this fact which permits the distinction between Kelvin-Planck systems and strong Kelvin-Planck systems; for whenever $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is closed, the strong Kelvin-Planck condition (E.26) reduces to

$$[\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)] \cap \mathscr{M}_+(\Sigma) = \{0\}.$$

This is readily shown to be equivalent to the Kelvin-Planck condition (E.2).

There are two easily described circumstances under which $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is closed. First, consider a cyclic heating system for which $-\mathscr{M}_+(\Sigma)$ is contained in $\hat{\mathscr{C}}$. This is to say that every negative measure is approximated by the cyclic heating measures (and their positive multiples).* Under these circumstances $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is identical to the closed set $\hat{\mathscr{C}}$. Moreover, we have

$$-\mathscr{M}_+(\Sigma) \subset \hat{\mathscr{C}} \Rightarrow \tilde{\mathscr{C}} = \hat{\mathscr{C}}.$$

Next, consider a Kelvin-Planck system for which $\Sigma$ is compact. Here again $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is closed, as we show in the following proposition.

**Proposition E.1.** *Let* $(\Sigma, \mathscr{C})$ *be a Kelvin-Planck system with compact* $\Sigma$. *Then* $(\Sigma, \mathscr{C})$ *is a strong Kelvin-Planck system. In fact,*

$$\tilde{\mathscr{C}} = \hat{\mathscr{C}} - \mathscr{M}_+(\Sigma).$$

*That is,* $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ *is closed.*

---

* The situation described is hardly far-fetched. In fact we *must* have $-\mathscr{M}_+(\Sigma) \subset \hat{\mathscr{C}}$ for any Kelvin-Planck system $(\Sigma, \mathscr{C})$ with compact $\Sigma$ which has at least one irreversible element and which admits an essentially unique Clausius temperature scale. See Corollary 9.2.

**Proof.** When $\Sigma$ is compact Theorem 4.1 ensures that the Kelvin-Planck system $(\Sigma, \mathscr{C})$ admits a Clausius temperature scale, whereupon Remark E.3 requires that $(\Sigma, \mathscr{C})$ be a strong Kelvin-Planck system. Next we show that $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is closed. Suppose that $\nu \in \mathscr{M}(\Sigma)$ is not in $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$. Then, with $\mathscr{K}(\nu)$ defined as in Lemma 6.1, it is easy to show that $\mathscr{K}(\nu)$ is disjoint from $\hat{\mathscr{C}}$. Thus, Lemma 6.1 ensures the existence of a Clausius scale $T$ such that

$$\int_\Sigma \frac{d\nu}{T} > 0.$$

But arguments in Remark E.3 taken together with (E.29) require that, for any Clausius scale $T$, we must have

$$\int_\Sigma \frac{d\mu}{T} \leqq 0, \qquad \forall \mu \in cl\,(\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)) = \tilde{\mathscr{C}}. \tag{E.30}$$

Thus, $\nu$ cannot lie in the closure of $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$. From this we conclude that $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is closed.

We know that every cyclic heating system that admits a Clausius temperature scale must be a strong Kelvin-Planck system. As we shall see, under a fairly mild additional condition the converse is true as well. The following lemma lays the groundwork for the theorem we wish to prove.

**Lemma E.1.** *Let $(\Sigma, \mathscr{C})$ be a strong Kelvin-Planck system. Then, for any compact $K \subset \Sigma$, there exists a non-negative function $\phi \in C(\Sigma, \mathbb{R})$ such that*

$$\int_\Sigma \phi \, d\varphi \leqq 0, \qquad \forall \varphi \in \hat{\mathscr{C}}$$

*and $\phi(\sigma) > 0$ for all $\sigma \in K$.*

**Proof.** Recall the notation given in (E.1), and see Theorem F.1 in Appendix F. Since $(\Sigma, \mathscr{C})$ is a strong Kelvin-Planck system we must have, for any compact $K \subset \Sigma$, that

$$\mathscr{M}_+^{K,1}(\Sigma) \cap cl\,[\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)]$$

is empty. Since $\mathscr{M}_+^{K,1}\Sigma)$ is compact and convex and since $\hat{\mathscr{C}}$ and $\mathscr{M}_+(\Sigma)$ are convex cones, Theorem F.1 ensures the existence of a continuous linear function $f: \mathscr{M}(\Sigma) \to \mathbb{R}$ such that

$$f(\nu) > 0, \qquad \forall \nu \in \mathscr{M}_+^{K,1}(\Sigma),$$

$$f(\varphi) \leqq 0, \qquad \forall \varphi \in \hat{\mathscr{C}}$$

and

$$f(\mu) \geqq 0, \qquad \forall \mu \in \mathscr{M}\ (\Sigma).$$

Moreover, since every continuous linear real-valued function on $\mathscr{M}(\Sigma)$ is of the form $\tilde{\phi}$ for some $\phi \in C(\Sigma, \mathbb{R})$ we have, for some $\phi \in C(\Sigma, \mathbb{R})$,

$$\int_\Sigma \phi \, d\nu > 0, \qquad \forall \nu \in \mathscr{M}_+^{K,1}(\Sigma), \tag{E.31}$$

$$\int_\Sigma \phi \, d\varphi \leqq 0, \qquad \forall \varphi \in \hat{\mathscr{C}} \tag{E.32}$$

and

$$\int_\Sigma \phi \, d\mu \geqq 0, \qquad \forall \mu \in \mathscr{M}_+(\Sigma). \tag{E.33}$$

Now, for each $\sigma \in \Sigma$, the Dirac measure $\delta_\sigma$ lies in $\mathscr{M}_+(\Sigma)$; thus, from (E.33) we obtain $\phi(\sigma) \geqq 0$ for all $\sigma \in \Sigma$. Moreover, for each $\sigma \in K$ we have that $\delta_\sigma$ lies in $\mathscr{M}_+^{K,1}(\Sigma)$; hence, from (E.31) we have $\phi(\sigma) > 0$ for all $\sigma \in K$. These positivity properties of $\phi$ taken together with (E.32) give the desired result.

The functions given by Lemma E.1 may fail to provide Clausius temperature scales only because each may vanish somewhere in $\Sigma$ and, therefore, fail to admit a reciprocal. We do, however, have for each compact $K \subset \Sigma$ a non-negative continuous function on $\Sigma$ which is positive on $K$ and integrates non-positively against all cyclic heating measures. This raises the possibility that such functions, corresponding to sufficiently many compact sets, can be "patched together" to ensure the existence of a continuous function which integrates non-negatively against every cyclic heating measure and is positive everywhere on $\Sigma$. Such a construction can, in fact, be effected when $\Sigma$ satisfies a mild topological condition.

Recall that a topological space is *$\sigma$-compact* if it is the union of countably many compact subsets. If the locally compact Hausdorff space $\Sigma$ is $\sigma$-compact then there exists a sequence $\{K_n\}$ of compact sets which cover $\Sigma$ and are such that

$$K_n \subset \text{interior}\,(K_{n+1}), \qquad n = 1, 2, \ldots$$

(See [B], p. 94.) Note that $\sigma$-compactness is substantially weaker than compactness. In particular, any locally compact space with a countable base of open sets is $\sigma$-compact. For example, *$\sigma$-compactness is a property of any locally compact subset of a finite-dimensional vector space (endowed with the usual topology).*

**Theorem E.1.** *Let $(\Sigma, \mathscr{C})$ be a cyclic heating system. Among the following statements we have the implication* (ii) $\Rightarrow$ (i), *and when $\Sigma$ is $\sigma$-compact,* (i) $\Rightarrow$ (ii). *In particular,* (i) *and* (ii) *are equivalent when $\Sigma$ has a countable base of open sets.*

(i) *$(\Sigma, \mathscr{C})$ is a strong Kelvin-Planck system.*

(ii) *There exists a continuous function $T: \Sigma \to \mathbb{P}$ such that*

$$\int_\Sigma \frac{d\varphi}{T} \leqq 0, \qquad \forall \varphi \in \mathscr{C}.$$

**Proof.** The implication (ii) $\Rightarrow$ (i) was already proved in Remark E.3. Thus we suppose that $\Sigma$ is $\sigma$-compact and prove the implication (i) $\Rightarrow$ (ii). Let $\{K_n\}$

be a sequence of compact sets in $\Sigma$ having the properties described above. For each $n$, Lemma E.1 provides $0 \leqq \phi_n \in C(\Sigma, \mathbb{R})$, with $\phi_n(\sigma) > 0$ for all $\sigma \in K_n$, such that

$$\int_\Sigma \phi_n \, d\varphi \leqq 0, \quad \forall \varphi \in \mathscr{C}. \tag{E.34}$$

Moreover, by suitable scaling if necessary, we may suppose that

$$\phi_n(\sigma) \leqq 2^{-n}, \quad \forall \sigma \in K_n. \tag{E.35}$$

Using (E.35), we may establish that

$$\sigma \in K_N, \; M > N \Rightarrow \sum_{n=M}^{\infty} \phi_n(\sigma) \leqq 2^{1-M}. \tag{E.36}$$

Since each $\sigma \in \Sigma$ resides in some $K_N$, (E.36) implies that the series

$$\sum_{n=1}^{\infty} \phi_n(\sigma)$$

converges for each $\sigma$. We define the function $\phi : \Sigma \to \mathbb{P}$ by

$$\phi = \sum_{n=1}^{\infty} \phi_n. \tag{E.37}$$

The implication (E.36) ensures that the series on (E.37) converges uniformly on each $K_N$. Thus, the restriction of $\phi$ to each $K_N$ is continuous. Now each $\sigma \in \Sigma$ lies in the interior of some $K_N$ so that $\phi$ is continuous at every $\sigma$. Thus, we have $\phi \in C(\Sigma, \mathbb{R})$. Since, for every $\sigma \in \Sigma$, $\phi_n(\sigma) > 0$ for some $n$ we also have that $\phi$ takes positive values everywhere on $\Sigma$. Now let $T(\cdot) = 1/\phi(\cdot)$. Then (E.34) and (E.37) give

$$\int_\Sigma \frac{d\varphi}{T} \leqq 0, \quad \forall \varphi \in \mathscr{C}.$$

The last assertion of the theorem results from the fact that if the locally compact space $\Sigma$ has a countable base of open sets then $\Sigma$ is $\sigma$-compact.

The following example demonstrates that when $\Sigma$ is not $\sigma$-compact even a strong Kelvin-Planck system may fail to admit a Clausius temperature scale.

**Example E.2.** Let $\Sigma$ be an uncountable set with the discrete topology. Then all functions on $\Sigma$ are continuous, and the compact sets in $\Sigma$ are just the finite ones. Let $\sigma_0$ be a fixed element of $\Sigma$ and let

$$\mathscr{C} = \{\varphi \in \mathscr{M}(\Sigma) \mid \varphi(\{\sigma_0\}) \leqq 0 \text{ and } \varphi(\{\sigma\}) + \varphi(\{\sigma_0\}) \leqq 0, \quad \forall \sigma \in \Sigma\}.$$

The set $\mathscr{C}$ is clearly a convex cone. Writing $\chi_\sigma$ for the (continuous) function on $\Sigma$ which takes the value one at $\sigma$ and zero elsewhere, we have for every $\mu \in \mathscr{M}(\Sigma)$

$$\int_\Sigma \chi_{\sigma_0} \, d\mu = \mu(\{\sigma_0\})$$

and, for each $\sigma \in \Sigma$,

$$\int_{\Sigma} (\chi_{\sigma_0} + \chi_\sigma)\, d\mu = \mu(\{\sigma_0\}) + \mu(\{\sigma\}).$$

Thus, functions in the set

$$\{\chi_{\sigma_0}\} \cup \{\chi_{\sigma_0} + \chi_\sigma\}_{\sigma \in \Sigma}$$

integrate non-positively against all measures in $\mathscr{C}$. On the other hand, any measure not in $\mathscr{C}$ will integrate at least one of these functions positively. From this it follows easily that $\mathscr{C}$ is closed. Thus, we have $\hat{\mathscr{C}} = \mathscr{C}$. Moreover, $\hat{\mathscr{C}}$ contains no non-zero positive measure so that $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system. In fact, $\hat{\mathscr{C}}$ contains $-\mathscr{M}_+(\Sigma)$, whereupon $(\Sigma, \mathscr{C})$ is a strong Kelvin-Planck system (Remark E.4).

Now we show that $(\Sigma, \mathscr{C})$ admits no Clausius temperature scale. Suppose that $T: \Sigma \to \mathbb{P}$ is such a scale. Choose a finite set $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ not containing $\sigma_0$ such that

$$\sum_{i=1}^{n} \frac{1}{T(\sigma_i)} > \frac{1}{T(\sigma_0)}. \tag{E.38}$$

This can be done because, for at least one positive integer $k$, the set

$$\{\sigma \in \Sigma \mid T(\sigma) \leqq k\} \tag{E.39}$$

is uncountable; with $k$ such an integer we can choose from (E.39) a sufficiently large finite number of elements as to satisfy the requirements of (E.38). Now the measure

$$q = -\delta_{\sigma_0} + \sum_{i=1}^{n} \delta_{\sigma_i}$$

clearly belongs to $\mathscr{C}$. Hence, for the Clausius scale $T$ we must have

$$\int_{\Sigma} \frac{dq}{T} = \left(\sum_{i=1}^{n} \frac{1}{T(\sigma_i)}\right) - \frac{1}{T(\sigma_0)} \leqq 0.$$

But this contradicts (E.38). Therefore, the strong Kelvin-Planck system $(\Sigma, \mathscr{C})$ admits no Clausius temperature scale.

Later, we shall formulate a condition stronger than (E.26) which, for a cyclic heating system $(\Sigma, \mathscr{C})$, is *equivalent* to the existence of a Clausius temperature scale for *any* locally compact Hausdorff $\Sigma$.

First, however, we take up the question of uniqueness of Clausius scales when they do exist. For compact $\Sigma$ this was settled by Theorem 9.1. Moreover, in Remark 9.2 we indicated that if $T$ is Clausius scale for a cyclic heating system $(\Sigma, \mathscr{C})$ with compact $\Sigma$, then all other Clausius scales are positive constant multiples of $T$ if and only if, for all $\sigma'$, $\sigma \in \Sigma$, both $T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma$ and its negative are members of $\hat{\mathscr{C}}$. When $\Sigma$ is not compact this last condition remains sufficient for uniqueness; but, as we show in the following example, it is no longer necessary.

**Example E.3.** Let $\Sigma = [1, \infty)$ and, for $x \in \Sigma$, let

$$\phi_1(x) = \frac{1}{x}, \quad \phi_2(x) = \frac{1}{x} - 1.$$

Moreover, let

$$\mathscr{F} := \{s\phi_1 + t\phi_2 \in C(\Sigma, \mathbb{R}) \mid s \geqq 0, t \geqq 0\}$$

and

$$\mathscr{C} := \left\{ \varphi \in \mathscr{M}(\Sigma) \,\middle|\, \int_\Sigma \phi \, d\varphi \leqq 0, \quad \forall \phi \in \mathscr{F} \right\}.$$

It is shown in Appendix G that $\mathscr{C}$ is a closed convex cone (i.e., $\mathscr{C} = \hat{\mathscr{C}}$) and that

$$\phi \in C(\Sigma, \mathbb{R}) \text{ and } \int_\Sigma \phi \, d\varphi \leqq 0, \; \forall \varphi \in \mathscr{C} \;\Rightarrow\; \phi \in \mathscr{F}. \tag{E.39}$$

Note that the function $T(\cdot) := 1/\phi_1(\cdot)$ is a Clausius scale for the cyclic heating system $(\Sigma, \mathscr{C})$. Note also that $\phi_1$ and its positive constant multiples are the only functions of $\mathscr{F}$ that take positive values everywhere on $\Sigma$. Thus, it follows from (E.39) that $T(\cdot)$ and its positive multiples are the only Clausius scales for $(\Sigma, \mathscr{C})$.

Now we show that, with $z > y$, the measure $T(y)\,\delta_y - T(z)\,\delta_z$ cannot be an element of $\hat{\mathscr{C}}$. Note that since $T(\cdot) = 1/\phi_1(\cdot)$ we have

$$T(y)\,\delta_y - T(z)\,\delta_z = y\,\delta_y - z\,\delta_z.$$

Integrating $\phi_2$ against this measure we obtain the positive number $(z - y)$; thus, the measure cannot lie in $\hat{\mathscr{C}}$. (On the other hand, it is readily confirmed that $T(z)\,\delta_z - T(y)\,\delta_y$ is an element of $\hat{\mathscr{C}}$.)

The following theorem generalizes uniqueness results of Section 9 to the situation in which $\Sigma$ is locally compact. The theorem asserts, in effect, that those results carry over from the compact case provided that $\tilde{\mathscr{C}}$ replaces $\hat{\mathscr{C}}$. Here $\sigma$-compactness plays no role.

**Theorem E.2.** *Suppose that* $T: \Sigma \to \mathbb{P}$ *is a Clausius temperature scale for a cyclic heating system* $(\Sigma, \mathscr{C})$ *and that* $\sigma$ *and* $\sigma'$ *are elements of* $\Sigma$. *Then the following are equivalent*:

(i) *Every Clausius scale* $T^\dagger: \Sigma \to \mathbb{P}$ *for* $(\Sigma, \mathscr{C})$ *satisfies*

$$\frac{T^\dagger(\sigma')}{T^\dagger(\sigma)} = \frac{T(\sigma')}{T(\sigma)}.$$

(ii) *Both* $T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma$ *and* $T(\sigma)\,\delta_\sigma - T(\sigma')\,\delta_{\sigma'}$ *are elements of* $\tilde{\mathscr{C}}$.

**Proof.** In Remark E.3 we showed that the reciprocal of every Clausius temperature scale for a cyclic heating system $(\Sigma, \mathscr{C})$ integrates non-positively against all measures in $\tilde{\mathscr{C}}$. To prove that (ii) $\Rightarrow$ (i) it is only necessary to use this fact and integrate $T^\dagger$ against both measures mentioned in (ii).

To prove that (i) $\Rightarrow$ (ii) we suppose that (i) holds but that the compact convex set consisting only of $T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma$ is disjoint from the closed convex cone

$$\tilde{\mathscr{C}} = cl\,(\text{Cone}\,(\mathscr{C}) - \mathscr{M}_+(\Sigma)).$$

Then Theorem 2.1 ensures the existence of $\phi \in C(\Sigma, \mathbb{R})$ such that

$$\int_\Sigma \phi\, d[T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma] = T(\sigma')\,\phi(\sigma') - T(\sigma)\,\phi(\sigma) > 0 \tag{E.40}$$

and

$$\int_\Sigma \phi\, d\mu \leqq 0, \quad \forall \mu \in \tilde{\mathscr{C}}. \tag{E.41}$$

Since, for every $\sigma'' \in \Sigma$, $-\delta_{\sigma''}$ is contained in $\tilde{\mathscr{C}}$ we obtain from (E.41) that $\phi$ is non-negative on $\Sigma$, and (E.40) requires that $\phi$ not be identically zero. Moreover, since $\mathscr{C}$ is contained in $\tilde{\mathscr{C}}$ we have, again from (E.41), that

$$\int_\Sigma \phi\, d\varphi \leqq 0, \quad \forall \varphi \in \mathscr{C}.$$

Thus the function $T^\dagger : \Sigma \to \mathbb{P}$ defined by

$$\frac{1}{T^\dagger(\cdot)} = \phi(\cdot) + \frac{1}{T(\cdot)} \tag{E.42}$$

is a Clausius scale different from $T$. In fact, with $T^\dagger$ taken as in (E.42) it follows from (E.40) that the equation in (i) cannot hold. This contradicts what has been supposed. Proof that $T(\sigma)\,\delta_\sigma - T(\sigma')\,\delta_{\sigma'}$ lies in $\tilde{\mathscr{C}}$ is similar.

As an easy consequence of Theorem E.2 we have:

**Corollary E.1.** *Suppose that* $T : \Sigma \to \mathbb{P}$ *is a Clausius temperature scale for a cyclic heating system* $(\Sigma, \mathscr{C})$. *Then the following statements are equivalent:*

(i) *Every Clausius temperature scale for* $(\Sigma, \mathscr{C})$ *is a positive constant multiple of* $T(\cdot)$.

(ii) *For all* $\sigma', \sigma \in \Sigma$ *both* $T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma$ *and* $T(\sigma)\,\delta_\sigma - T(\sigma')\,\delta_{\sigma'}$ *are elements of* $\tilde{\mathscr{C}}$.

By setting $T(\sigma') = T(\sigma)$ in Theorem E.2 we obtain a generalization of Theorem 6.1:

**Corollary E.2.** *Let* $(\Sigma, \mathscr{C})$ *be a cyclic heating system that admits at least one Clausius temperature scale, and let* $\sigma'$ *and* $\sigma$ *be elements of* $\Sigma$. *The following statements are equivalent:*

(i) $T(\sigma') = T(\sigma)$ *for every Clausius scale* $T$.

(ii) *Both* $\delta_{\sigma'} - \delta_\sigma$ *and* $\delta_\sigma - \delta_{\sigma'}$ *are elements of* $\tilde{\mathscr{C}}$.

**Remark E.4.** *If the cyclic heating system* $(\Sigma, \mathscr{C})$ *of Theorem E.2 is such that* $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ *is closed* (i.e. if $\tilde{\mathscr{C}} = \hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$), *then condition* (ii) *of Theorem E.2* *can be satisfied only if the two measures mentioned there lie in* $\hat{\mathscr{C}}$. (If $\Sigma$ is compact, Proposition E.1 asserts that $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is closed so that the results given by Theorem E.2 and its corollaries reduce, as they should, to those given in the main body of this article. We are claiming here that this same reduction takes place whenever $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is closed, even if $\Sigma$ is not compact.)

**Proof.** Suppose that $\tilde{\mathscr{C}} = \hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ and that condition (ii) is satisfied. Then

$$T(\sigma')\,\delta_{\sigma'} - T(\sigma)\,\delta_\sigma = \varphi_1 - \nu_1$$

and

$$T(\sigma)\,\delta_\sigma - T(\sigma')\,\delta_{\sigma'} = \varphi_2 - \nu_2,$$

where $\varphi_1$ and $\varphi_2$ are (non-zero) elements of $\hat{\mathscr{C}}$ and $\nu_1$ and $\nu_2$ are elements of $\mathscr{M}_+(\Sigma)$. Adding these equations and rearranging, we obtain

$$\varphi_1 + \varphi_2 = \nu_1 + \nu_2 \in \hat{\mathscr{C}} \cap \mathscr{M}_+(\Sigma).$$

By supposition the system $(\Sigma, \mathscr{C})$ admits a Clausius scale and must therefore have the (strong) Kelvin-Planck property. Hence,

$$\varphi_1 + \varphi_2 = \nu_1 + \nu_2 = 0.$$

Since $\nu_1$ and $\nu_2$ are elements of $\mathscr{M}_+(\Sigma)$, it can only be the case that $\nu_1 = 0$ and $\nu_2 = 0$.

**Remark E.5.** Even when $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ is not closed we may nevertheless assert that, if both measures mentioned in condition (ii) of Theorem E.2 lie in $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$, then both must in fact lie in $\hat{\mathscr{C}}$. (The same is true of any pair consisting of a measure and its negative.) Proof of this assertion is virtually the same as that given in the preceding remark. Note that if condition (ii) is satisfied, the two measures mentioned there can *fail* to lie in $\hat{\mathscr{C}}$ only if at least one of them lies in $cl\,(\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma))$ but not in $\hat{\mathscr{C}} - \mathscr{M}_+(\Sigma)$ itself.

We devote the rest of this appendix to further consideration of conditions on a cyclic heating system $(\Sigma, \mathscr{C})$ sufficient for the existence of a Clausius temperature scale. Although we believe that, from a practical standpoint, Theorem E.1 resolves this issue for thermodynamical theories in which the state space is locally compact, we would nevertheless like to say something about cyclic heating systems for which $\Sigma$ is locally compact but not $\sigma$-compact. Recall that for such systems the strong Kelvin-Planck property (E.26) is necessary but not sufficient for the existence of a Clausius temperature scale.

For a Kelvin-Planck system $(\Sigma, \mathscr{C})$ it must be the case that

$$cl\,[\text{Cone}\,(\mathscr{C})] \cap \mathscr{M}^1_+(\Sigma) = \emptyset.$$

That is, for each $\nu \in \mathscr{M}^1_+(\Sigma)$ there must exist an open neighborhood of $\nu$, say $\mathscr{N}_\nu$, which is disjoint from Cone $(\mathscr{C})$. Moreover, because $\mathscr{M}(\Sigma)$ is locally convex we can assume that each $\mathscr{N}_\nu$ is convex. By taking the union of all $\mathscr{N}_\nu$ we obtain an open set which is a neighborhood of every $\nu \in \mathscr{M}^1_+(\Sigma)$ and which is disjoint from Cone ($\mathscr{C}$). We cannot, however, assert that the open set so obtained is convex. Even though there exists for each element of $\mathscr{M}^1_+(\Sigma)$ an open convex neighborhood disjoint from Cone ($\mathscr{C}$), there may exist no single open *convex* set which is disjoint from Cone ($\mathscr{C}$) and is a neighborhood of *every* $\nu \in \mathscr{M}^1_+(\Sigma)$. If such a set exists we say that the Kelvin-Planck system $(\Sigma, \mathscr{C})$ is *uniform.*

**Definition E.2.** *A* **uniform Kelvin-Planck system** *is a cyclic heating system* $(\Sigma, \mathscr{C})$ *with the following property: These exists a convex open neighborhood of* $\mathscr{M}^1_+(\Sigma)$ *which is disjoint from* Cone ($\mathscr{C}$).

**Theorem E.3.** *For a cyclic heating system* $(\Sigma, \mathscr{C})$ *the following statements are equivalent:*

(i) $(\Sigma, \mathscr{C})$ *is a uniform Kelvin-Planck system.*

(ii) *There exists a continuous function* $T: \Sigma \to \mathbb{P}$ *such that*

$$\int_\Sigma \frac{d\varphi}{T} \leqq 0, \qquad \forall \varphi \in \mathscr{C}.$$

**Proof.** In proving that (i) implies (ii), we suppose that $\mathscr{N} \subset \mathscr{M}(\Sigma)$ is a convex open neighborhood of $\mathscr{M}^1_+(\Sigma)$ which is disjoint from Cone ($\mathscr{C}$). It is easy to show that $\mathscr{N}$ is also disjoint from the convex cone $\hat{\mathscr{C}} := cl$ [Cone ($\mathscr{C}$)]. Now we invoke another version of the Hahn-Banach Theorem ([C1], Theorem 21.11) which asserts that in a topological vector space two disjoint non-empty convex sets, at least one of which is open, admit separation by a closed hyperplane. In our context, this theorem implies that there exist $\phi \in C(\Sigma, \mathbb{R})$ and $\gamma \in \mathbb{R}$ such that

$$\int_\Sigma \phi \, d\nu > \gamma, \qquad \forall \nu \in \mathscr{N} \tag{E.43}$$

and

$$\int_\Sigma \phi \, d\varphi \leqq \gamma, \qquad \forall \varphi \in \hat{\mathscr{C}}. \tag{E.44}$$

Since $\hat{\mathscr{C}}$ is a cone, it follows easily from (E.44) that $\gamma$ cannot be negative. Thus, we have

$$\int_\Sigma \phi \, d\nu > 0, \qquad \forall \nu \in \mathscr{N} \tag{E.45}$$

and

$$\int_\Sigma \phi \, d\varphi \leqq 0, \qquad \forall \varphi \in \hat{\mathscr{C}}. \tag{E.46}$$

Because, for every $\sigma \in \Sigma$, the Dirac measure $\delta_\sigma$ lies in $\mathscr{N}$, we obtain from (E.45) that

$$\phi(\sigma) = \int_\Sigma \phi \, d\delta_\sigma > 0, \qquad \forall \sigma \in \Sigma.$$

Now (ii) emerges from (E.46) by setting $T(\cdot) = 1/\phi(\cdot)$.

To prove that (ii) implies (i) we need only observe that $\mathcal{M}^1_+(\Sigma)$ lies in the (convex) open half-space

$$\left\{\nu \in \mathcal{M}(\Sigma) \,\middle|\, \int_\Sigma \frac{d\nu}{T} > 0\right\}.$$

This half-space is clearly disjoint from Cone ($\mathscr{C}$), for each element of Cone ($\mathscr{C}$) integrates $1/T(\cdot)$ to a non-positive number.

**Corollary E.3.** *Every uniform Kelvin-Planck system is a strong Kelvin-Planck system. Moreover, every strong Kelvin-Planck system with a $\sigma$-compact state space is a uniform Kelvin-Planck system.*

**Proof.** If $(\Sigma, \mathscr{C})$ is a uniform Kelvin-Planck system Theorem E.3 ensures the existence of a Clausius temperature scale for it. In this case, Theorem E.1 requires that $(\Sigma, \mathscr{C})$ be a strong Kelvin-Planck system. Similarly if $(\Sigma, \mathscr{C})$ is a strong Kelvin-Planck system for which $\Sigma$ is $\sigma$-compact, Theorem E.1 ensures the existence of a Clausius scale for $(\Sigma, \mathscr{C})$. But then Theorem E.3 requires that $(\Sigma, \mathscr{C})$ be a uniform Kelvin-Planck system.

**Remark E.6.** A strong Kelvin-Planck system $(\Sigma, \mathscr{C})$ for which $\Sigma$ is not $\sigma$-compact need not be a uniform Kelvin-Planck system. In fact, the strong Kelvin-Planck system described in Example E.2 admits no Clausius scale and therefore cannot be a uniform Kelvin-Planck system. *Thus, the uniform Kelvin-Planck condition described in Definition E.2 is more stringent than the strong Kelvin-Planck condition described in Definition E.1.*

Despite the fact that the uniform and strong Kelvin-Planck conditions coincide for $\sigma$-compact locally compact state spaces and despite the fact that all uniform Kelvin-Planck systems admit Clausius scales, we think the uniform Kelvin-Planck condition is somewhat less appealing than the three conditions (E.13)–(E.15) discussed earlier. Those resulted merely from the taking of closures in the three natural and equivalent conditions (E.10)–(E.12), the idea being that cyclic processes should not come arbitrarily close to standing in violation of the Kelvin-Planck Second Law. Where the Kelvin-Planck condition (E.14) requires that $\mathcal{M}^1_+(\Sigma)$ have an open neighborhood disjoint from Cone ($\mathscr{C}$), the uniform Kelvin-Planck condition requires that this neighborhood should also be *convex*. Although there may be a compelling argument for the imposition of this additional stricture, it is not one we have been able to make.

**Remark E.7.** In our preliminary study [FL] we confined our attention to the situation in which elements of $\Sigma$ are identified with readings on a presupposed empirical temperature scale. That is, $\Sigma$ was taken to be an interval of the real line. Our effort was devoted almost entirely to study of cyclic processes in which no material point experiences an empirical temperature outside a fixed closed and bounded set. Thus, for most of our preliminary work $\Sigma$ was identified with a compact interval. By invoking the Kelvin-Planck condition, we were able to prove the existence of a Clausius temperature scale much as we did in Theorem 4.1. Furthermore, we asserted in [FL] that when $\Sigma$ is an open (possibly unbounded) interval the existence of a Clausius temperature scale follows from the more restrictive uniform Kelvin-Planck condition. Working independently with heating measures defined on an open interval of empirical temperatures, ŠILHAVÝ [S4] obtained the same result by invoking the uniform Kelvin-Planck condition in a different but equivalent form: *There exists an open convex cone which contains the non-zero positive measures but no cyclic heating measure.* Since intervals of the real line are $\sigma$-compact, both we and ŠILHAVÝ could just as well have employed the strong Kelvin-Planck condition in dealing with non-compact intervals of empirical temperature. Insofar as the uniform Kelvin-Planck condition invokes rather arbitrarily the existence of *convex* open sets with prescribed properties, we think the strong Kelvin-Planck condition is to be preferred.

We close this appendix by indicating why it is that classical arguments *seem* to deliver Clausius temperature scales for non-compact state spaces on the basis of the Kelvin-Planck condition rather than the more stringent strong or uniform Kelvin-Planck conditions. The fact of the matter is that the classical arguments do *not* provide the existence of Clausius scales *solely* on the basis of the Kelvin-Planck condition, *for they also invoke the existence of a large supply of reversible cycles (and Carnot cycles in particular).*

Let $(\Sigma, \mathscr{C})$ be a Kelvin-Planck system with $\Sigma$ locally compact and Hausdorff. For each compact $K \subset \Sigma$ let $\mathscr{C}_K \subset \mathscr{M}(K)$ be the closed convex cone obtained by taking the restriction of measures in

$$\hat{\mathscr{C}} \cap \mathscr{M}^K(\Sigma)$$

to Borel sets contained in $K$.* From the fact that $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system it follows that $(K, \mathscr{C}_K)$ is also a Kelvin-Planck system. Thus, Theorem 4.1 ensures that there exists a *Clausius temperature scale on $K$* — that is, a continuous function $\Theta_K : K \to \mathbb{P}$ such that

$$\int_\Sigma \frac{d\varphi}{\Theta_K} \leqq 0, \qquad \forall \varphi \in \mathscr{C}_K. \tag{E.47}$$

Moreover, *in the presence of a suitably large supply of Carnot elements for $(K, \mathscr{C}_K)$,* Theorem 9.1 asserts that $\Theta_K(\cdot)$ is *essentially unique*—that is, that all positive continuous functions on $K$ satisfying (E.47) are constant multiples of $\Theta_K(\cdot)$. The existence of an essentially unique Clausius scale on each compact $K \subset \Sigma$ ensures the existence of a Clausius scale for $(\Sigma, \mathscr{C})$. This we show in the following proposition.

**Proposition E.2.** *Let $\Sigma$ be a locally compact Hausdorff space, and let $(\Sigma, \mathscr{C})$ be a cyclic heating system. If, for every compact $K \subset \Sigma$, there is an essentially unique Clausius temperature scale on $K$, then $(\Sigma, \mathscr{C})$ admits an (essentially unique) Clausius temperature scale.*

**Proof.** Let $\sigma_0$ be some fixed element of $\Sigma$. If $K$ is any compact set containing $\sigma_0$ we denote by $\Theta_K$ the (unique) Clausius scale on $K$ that takes the value one at

* The linear subspace $\mathscr{M}^K(\Sigma) \subset \mathscr{M}(\Sigma)$ defined in (E.1) is closed.

$\sigma_0$. Note that if $K$ and $K'$ are compact sets in $\Sigma$ such that $\sigma_0 \in K \subset K'$, then $\Theta_K$ is the restriction of $\Theta_{K'}$ to $K$.

Now we let $T: \Sigma \to \mathbb{P}$ be defined as follows: For each $\sigma \in \Sigma$ set $T(\sigma) = \Theta_K(\sigma)$ for any compact $K$ containing both $\sigma$ and $\sigma_0$. There is no ambiguity here, for if $K'$ is a different compact set containing $\sigma$ and $\sigma_0$ we have $\Theta_K(\sigma) = \Theta_{K \cup K'}(\sigma) = \Theta_{K'}(\sigma)$. Note that for any compact $K$ containing $\sigma_0$, $T$ extends $\Theta_K$. The function $T$ is continuous because each element $\sigma \in \Sigma$ has a compact neighborhood $N$ containing $\sigma_0$,* and on this neighborhood $T$ agrees with $\Theta_N \in C(N, \mathbb{R})$.

Next we show that $T$ is a Clausius scale for $(\Sigma, \mathscr{C})$. Let $\varphi$ be any element of $\mathscr{C}$, and suppose that the support of $\varphi$ lies in some compact $K \subset \Sigma$ containing $\sigma_0$.** Furthermore, let $\bar{\varphi}$ denote the restriction of $\varphi$ to those Borel sets of $\Sigma$ contained in $K$. Thus, $\bar{\varphi}$ is an element of $\mathscr{C}_K$. Then we have

$$\int_\Sigma \frac{d\varphi}{T} = \int_K \frac{d\varphi}{T} = \int_K \frac{d\bar{\varphi}}{\Theta_K} \leqq 0.$$

Therefore, $T: \Sigma \to \mathbb{P}$ is a Clausius scale for $(\Sigma, \mathscr{C})$.

The essential uniqueness of the scale $T$ follows from the fact that any Clausius scale for $(\Sigma, \mathscr{C})$ which takes the value one at $\sigma_0$, when restricted to any compact $K \subset \Sigma$ containing $\sigma_0$, must by hypothesis coincide with $\Theta_K$.

**Remark E.8.** For a Kelvin-Planck system $(\Sigma, \mathscr{C})$, the existence of a Clausius temperature scale on each compact $K \subset \Sigma$ does not *by itself* ensure the existence of a Clausius temperature scale for $(\Sigma, \mathscr{C})$. In Example E.1 it is not difficult to see that, for any compact $K \subset [0, \infty)$, there exists a function $f_K \in \mathscr{F}$ that is positive on $K$. The function $\Theta_K: K \to \mathbb{P}$ defined by

$$\Theta_K(x) = 1/f_K(x), \quad \forall x \in K,$$

is a Clausius scale for the Kelvin-Planck system $(K, \mathscr{C}_K)$. Nevertheless, the Kelvin-Planck system $(\Sigma, \mathscr{C})$ of Example E.1 admits no Clausius temperature scale. It is the lack of an *essentially unique* Clausius scale on every compact $K \subset [0, \infty)$ that precludes application of Proposition E.2.

## Appendix F. A Variant of Theorem 2.1

There is a variant of Theorem 2.1 which, when applied to a Kelvin-Planck system $(\Sigma, \mathscr{C})$, will give conditions on $\mathscr{C}$ equivalent to the existence of Clausius temperature scales having special properties.

**Theorem F.1.** *Let $V$ be a Hausdorff locally convex topological vector space, let $K$ be a compact convex subset of $V$, and let $A$ and $B$ be convex cones in $V$. Then the following statements are equivalent:*

---

* Since $\Sigma$ is locally compact, $\sigma$ has a neighborhood $N'$ with compact closure. If $cl(N')$ does not contain $\sigma_0$, take $N = cl(N') \cup \{\sigma_0\}$.

** For example, take $K = \text{supp}(\varphi) \cup \{\sigma_0\}$.

(i) $K \cap cl\,(A - B)$ *is empty.*
(ii) *There exists a continuous linear function* $f: V \to \mathbb{R}$ *such that*

$$f(k) > 0 \text{ for all } k \in K,$$

$$f(a) \leqq 0 \text{ for all } a \in A$$

*and*

$$f(b) \geqq 0 \text{ for all } b \in B.$$

**Proof.** Since $cl\,(A - B)$ is a closed convex cone, condition (i) implies, by Theorem 2.1, the existence of a continuous linear function $f: V \to \mathbb{R}$ such that $f(k) > 0$ for all $k \in K$ and $f(c) \leqq 0$ for all $c \in cl\,(A - B)$. Since $A$ and $B$ are cones it follows easily that $f(a) \leqq 0$ for all $a \in A$ and $f(b) \geqq 0$ for all $b \in B$. Thus (i) implies (ii).

Now suppose there exists a continuous $f: V \to \mathbb{R}$ that satisfies the requirements of (ii). For any $a \in A$ and $b \in B$ we have

$$f(a - b) = f(a) - f(b) \leqq 0.$$

Thus for any $c \in A - B$ we have $f(c) \leqq 0$, and the continuity of $f$ ensures that $f(c) \leqq 0$ for any $c \in cl\,(A - B)$. Since for any $k$ in $K$ we have $f(k) > 0$ it follows that $K$ and $cl\,(A - B)$ can have no element in common. Hence (ii) implies (i).

Theorem F.1 may be applied to deduce a useful fact about Kelvin-Planck systems. Here, as in the main body of this article, $\Sigma$ is taken to be compact.

**Corollary F.1.** *Let* $(\Sigma, \mathscr{C})$ *be a Kelvin-Planck system, and let* $\mathscr{B} \subset \mathscr{M}(\Sigma)$ *be a convex cone. Then the following are equivalent:*
(i) $\mathscr{M}^1_+(\Sigma) \cap cl\,(\hat{\mathscr{C}} - \mathscr{B})$ *is empty.*
(ii) $[\mathscr{M}_+(\Sigma) \setminus \{0\}] \cap cl\,(\hat{\mathscr{C}} - \mathscr{B})$ *is empty.*
(iii) *There exists a Clausius temperature scale* $T: \Sigma \to \mathbb{P}$ *such that*

$$\int_\Sigma \frac{d\ell}{T} \geqq 0 \quad \text{for all} \quad \ell \in \mathscr{B}.$$

**Proof.** The implications (i) $\Leftrightarrow$ (ii) follow easily from the fact that both $[\mathscr{M}_+(\Sigma) \setminus \{0\})$ and $cl\,(\hat{\mathscr{C}} - \mathscr{B})$ are cones and the fact that

$$[\mathscr{M}_+(\Sigma) \setminus \{0\}] = \text{Cone}\,[\mathscr{M}^1_+(\Sigma)].$$

The implications (i) $\Leftrightarrow$ (iii) are proved by setting $V = \mathscr{M}(\Sigma)$, $K = \mathscr{M}^1_+(\Sigma)$, $A = \hat{\mathscr{C}}$ and $B = \mathscr{B}$ in Theorem F.1 and drawing on the fact that a linear function $f: \mathscr{M}(\Sigma) \to \mathbb{R}$ is continuous if and only if there exists a continuous $\phi: \Sigma \to \mathbb{R}$ such that

$$f(\mu) \equiv \int_\Sigma \phi \, d\mu.$$

The function $\phi(\cdot)$ is identified with $1/T(\cdot)$.

## Appendix G. On the Construction of Counterexamples

There are several instances in which we shall want to give counterexamples to assertions that cyclic heating systems with specified properties admit Clausius temperature scales of a certain kind. In each case we shall construct a cyclic heating system $(\Sigma, \mathscr{C})$, with $\mathscr{C}$ a closed convex cone in $\mathscr{M}(\Sigma)$, for which we know all elements of $C(\Sigma, \mathbb{R})$ that integrate nonpositively against every measure in $\mathscr{C}$. For this purpose we shall generally specify a closed convex cone $\mathscr{F} \subset C(\Sigma, \mathbb{R})$, define $\mathscr{C} \subset \mathscr{M}(\Sigma)$ by

$$\mathscr{C} := \left\{ q \in \mathscr{M}(\Sigma) \,\middle|\, \int_\Sigma f \, dq \leqq 0, \quad \forall f \in \mathscr{F} \right\},$$

and then assert that any element of $C(\Sigma, \mathbb{R})$ that integrates non-positively against each element of $\mathscr{C}$ must in fact be a member of $\mathscr{F}$. Here we provide the theoretical basis for constructions of this kind.

The facts we need can be stated in terms of a general locally convex Hausdorff topological vector space $V$. We denote by $V'$ the vector space of continuous real-valued linear functions on $V$, and we give $V'$ the weak-star topology—that is, the coarsest topology that, for every $v \in V$, renders continuous the function $f_v : V' \to \mathbb{R}$ defined by

$$f_v(v') \equiv v'(v).$$

With this topology, $V'$ is a locally convex Hausdorff topological vector space, and every real-valued continuous linear function on $V'$ is of the form $f_v(\cdot)$ for some $v \in V$ ([R2], p. 66).

**Proposition G.1.** *Let $V$ be a locally convex topological vector space, let $\mathscr{F}$ be a subset of $V$, and let*

$$\mathscr{F}^- := \{v' \in V' \mid v'(v) \leqq 0, \quad \forall v \in \mathscr{F}\}.$$

*Then*

(a) *$\mathscr{F}^-$ is a closed convex cone in $V'$,*

*and*

(b) *if $\mathscr{F}$ is a closed convex cone in $V$,*

$$v'(v) \leqq 0, \ \forall v' \in \mathscr{F}^- \ \Rightarrow \ v \in \mathscr{F}.$$

**Proof.** (a) For each $v \in V$ the half-space

$$\{v' \in V' \mid v'(v) \leqq 0\}$$

is a closed convex cone in $V'$. By taking the intersection of those half-spaces corresponding to all $v \in \mathscr{F}$, we obtain the set $\mathscr{F}^- \subset V'$. Since $\mathscr{F}^-$ is the intersection of closed convex cones, $\mathscr{F}^-$ must also be a closed convex cone.

(b) Suppose that $\mathscr{F}$ is a closed convex cone and that, for some $\bar{v} \in V$, we have

$$v'(\bar{v}) \leqq 0, \quad \forall v' \in \mathscr{F}^-. \tag{G.1}$$

Suppose further that $\bar{v}$ is not a member of $\mathscr{F}$. Since the set $\{\bar{v}\}$ is compact and convex, Theorem 2.1 ensures the existence of $v_0' \in V'$ such that

$$v_0'(\bar{v}) > 0 \tag{G.2}$$

and

$$v_0'(v) \leqq 0, \quad \forall v \in \mathscr{F}. \tag{G.3}$$

Now (G.3) asserts that $v_0'$ is a member of $\mathscr{F}^-$. But then (G.2) contradicts (G.1). Thus (G.1) requires that $\bar{v}$ be a member of $\mathscr{F}$.

The following proposition provides a simple way to construct closed convex cones in $V$.

**Proposition G.2.** *Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent set of vectors in $V$. Then the set*

$$P(v_1, v_2, \ldots, v_n) := \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \in V \mid \alpha_i \geqq 0, \quad i = 1, 2, \ldots, n\}$$

*is a closed convex cone.*

**Proof.** Straightforward computation shows that $P(v_1, v_2, \ldots, v_n)$ is a convex cone. Moreover, $P(v_1, v_2, \ldots, v_n)$ is a subset of the finite-dimensional (and therefore closed) linear subspace $S \subset V$ spanned by $\{v_1, v_2, \ldots, v_n\}$. Under the homeomorphism $T: \mathbb{R}^n \to S$ given by

$$T(x_1, x_2, \ldots, x_n) \equiv x_1 v_1 + x_2 v_2 + \ldots + x_n v_n,$$

the cone $P(v_1, v_2, \ldots, v_n)$ is the image of the closed set

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqq 0, \quad i = 1, 2, \ldots, n\}.$$

Therfore, $P(v_1, v_2, \ldots, v_n)$ is closed in the relative topology on $S$. Since $S$ is a closed subset of $V$, $P(v_1, v_2, \ldots, v_n)$ is a closed subset of $V$.

The following corollary to Proposition G.1 provides a way to construct cyclic heating systems for which all possible Clausius temperature scales (if any) are prescribed in advance. In the statement of the corollary it will be understood that $C(\Sigma, \mathbb{R})$ is given the usual supremum norm topology.

**Corollary G.1.** *Suppose that $\Sigma$ is a compact Hausdorff space and that $\mathscr{F}$ is a closed convex cone in $C(\Sigma, \mathbb{R})$. Moreover, let $\mathscr{C} \subset \mathscr{M}(\Sigma)$ be defined by*

$$\mathscr{C} := \mathscr{F}^- = \left\{ \varphi \in \mathscr{M}(\Sigma) \,\middle|\, \int_\Sigma f \, d\varphi \leqq 0, \quad \forall f \in \mathscr{F} \right\}.$$

*Then $\mathscr{C}$ is a closed convex cone in $\mathscr{M}(\Sigma)$; i.e., $\hat{\mathscr{C}} = \mathscr{C}$. If $\mathscr{F}$ contains a positive function, then $(\Sigma, \mathscr{C})$ is a Kelvin-Planck system. If $f \in C(\Sigma, \mathbb{R})$ satisfies*

$$\int_\Sigma f \, d\varphi \leqq 0, \quad \forall \varphi \in \mathscr{C},$$

*then $f$ is a member of $\mathscr{F}$. In particular, if $T: \Sigma \to \mathbb{P}$ is a Clausius temperature scale for $(\Sigma, \mathscr{C})$ then, for some $f \in \mathscr{F}$, $T(\cdot) = 1/f(\cdot)$.*

**Proof.** Take $V = C(\Sigma, \mathbb{R})$ and $V' = \mathcal{M}(\Sigma)$ in Proposition G.1. Using the definition of a Clausius temperature scale and Theorem 4.1, we obtain the desired result.

**Remark G.1.** In Appendix E we consider the case in which $\Sigma$ is locally compact but not necessarily compact. In this case the conclusions of Corollary G.1 still hold with $\mathcal{M}(\Sigma)$ interpreted as in Appendix E and with $C(\Sigma, \mathbb{R})$ given the *compact open topology*. This is the coarsest topology on $C(\Sigma, \mathbb{R})$ that renders open every set of the form

$$N_\phi(K, \alpha) := \{\phi' \in C(\Sigma, \mathbb{R}) : |\phi'(\sigma) - \phi(\sigma)| < \alpha, \quad \forall \sigma \in K\},$$

where $\phi$ is a member of $C(\Sigma, \mathbb{R})$, $\alpha > 0$, and $K \subset \Sigma$ is compact. Corollary G.1 then results from taking $V = C(\Sigma, \mathbb{R})$ and $V' = \mathcal{M}(\Sigma)$ in Proposition G.1, from the definition of a Clausius scale given near (E.4), and from Theorem E.1.

*Acknowledgment.* Once again we wish to aknowledge the influence JAMES SERRIN has had upon our work. For helpful conversations we thank PAUL BERNER, BERNARD COLEMAN, CHARLES CONLEY, LEOPOLDO NACHBIN, DAVID OWEN, and DAVID RUMSCHITZKI.

## References

[B] BOURBAKI, N., General Topology, Parts 1 and 2, Paris: Hermann, 1966. (All page numbers cited refer to Part 1 unless otherwise specified.)

[C1] CHOQUET, G., Lectures on Analysis, Vols. I–III, (eds. J. Marsden, T. Lance and S. Gelbart) New York: W. A. Benjamin Inc., 1969.

[C2] COLEMAN, B. D., Thermodynamics of materials with memory, Arch. Rational Mech. Anal., **17**, 1 (1964).

[CN] COLEMAN, B. D., & W. NOLL, The thermodynamics of elastic materials with heat conduction and viscosity, Arch. Rational Mech. Anal., **13**, 167 (1963).

[CO] COLEMAN, B. D., & D. R. OWEN, A mathematical foundation for thermodynamics, Arch. Rational Mech. Anal., **54**, 1 (1974).

[COS] COLEMAN, B. D., D. R. OWEN & J. B. SERRIN, The second law of thermodynamics for systems with approximate cycles, Arch. Rational Mech. Anal., **77**, 103 (1981).

[D1] DAY, W. A., Thermodynamics based on a work axiom, Arch. Rational Mech. Anal., **31**, 1 (1968).

[D2] DENBIGH, K., The Principles of Chemical Equilibrium, 3rd ed., Cambridge University Press, Cambridge, 1971.

[D3] DEBREU, F., Representation of a preference ordering by a numerical function, pp. 159–165 of Decision Processes, eds. R. M. Thrall, C. H. Coombs and R. L. Davis, Wiley, New York, 1954.

[D4] DIEUDONNÉ, J., Foundations of Modern Analysis, Academic Press, New York, 1969.

[F] FEYNMAN, R., The Character of Physical Law. Cambridge, Mass.: MIT Press, 1965.

[FL] FEINBERG, M., & R. LAVINE, Preliminary notes on the second law of thermodynamics, July 2, 1978.

[FS] FOSDICK, R. L., & J. SERRIN, Global properties of continuum thermodynamic processes. Arch. Rational Mech. Anal., **59**, 97 (1975).

[G] Gibbs, J. W., On the equilibrium of heterogeneous substances [Trans. Conn. Acad., **3**, 108–248, 343–524 (1875–1878)]. The Scientific Papers of J. Willard Gibbs, Vol. 1, 55–353. New York: Dover Publications, 1961.

[GW] Gurtin, M. E., & W. O. Williams, An axiomatic foundation for continuum thermodynamics. Arch. Rational Mech. Anal., **26**, 83 (1967).

[K] Kelley, J. L., General Topology. New York: D. Van Nostrand, 1955.

[KN] Kelley, J. L., & I. Namioka, Linear Topological Spaces. Berlin Heidelberg New York: Springer 1963.

[N1] Noll, W., Lectures on the foundations of continuum mechanics and thermodynamics. Arch. Rational Mech. Anal., **52**, 62 (1973).

[N2] Noll, W., On certain convex sets of measures and on phases of reacting mixtures. Arch. Rational Mech. Anal., **38**, 1 (1970).

[O] Owen, David R., The second law of thermodynamics for semi-systems with few approximate cycles, Arch. Rational Mech. Anal., **80**, 39 (1982).

[P] Pitteri, M., Classical thermodynamics of homogeneous systems based upon Carnot's general axiom, Arch. Rational Mech. Anal., **80**, 333 (1982).

[R1] Rudin, W., Real and Complex Analysis, 2nd ed. New York: McGraw-Hill, 1974.

[R2] Rudin, W., Functional Analysis. New York: McGraw-Hill, 1973.

[S1] Serrin, J., The concepts of thermodynamics, in Contemporary Developments in Continuum Mechanics and Partial Differential Equations (eds. G. de la Penha and L. A. J. Medeiros). Amsterdam: North-Holland Publishing Co. 1978.

[S2] Serrin, J., Conceptual analysis of the classical second laws of thermodynamics. Arch. Rational Mech. Anal., **70**, 355 (1979).

[S3] Serrin, J., Mathematical Foundations of Thermodynamics, notes of lectures delivered at the University of Naples, summer, 1980.

[S4] Šilhavý, M., On measures, convex cones, and foundations of thermodynamics I. Systems with vector-valued actions, II. Thermodynamic systems, Czechoslovak Journal of Physics, **B 30**, 841–861 and 961–991 (1980).

[T1] Truesdell, C., The Tragicomical History of Thermodynamics 1822–1854. Berlin Heidelberg New York: Springer 1980.

[T2] Truesdell, C., Absolute temperatures as a consequence of Carnot's general axiom. Arch. History of Exact Sciences, **20**, 357 (1979).

[T3] Truesdell, C., Rational Thermodynamics, 2nd ed., in press.

[TB] Truesdell, C., & S. Bharatha, The Concepts and Logic of Classical Thermodynamics as a Theory of Heat Engines. Berlin Heidelberg New York: Springer 1977.

Department of Chemical Engineering
University of Rochester
Rochester, New York

and

Department of Mathematics
University of Rochester
Rochester, New York

*(Received November 30, 1981)*