2H Mathematical Physics — Michaelmas Term

Supplement Sheet 2: The Cube

Consider a uniform cube of mass M of side length L. Let us consider the Moment of Inertia tensor $\mathbf{I}$ in coordinates with the origin at one corner. See the figure 1.

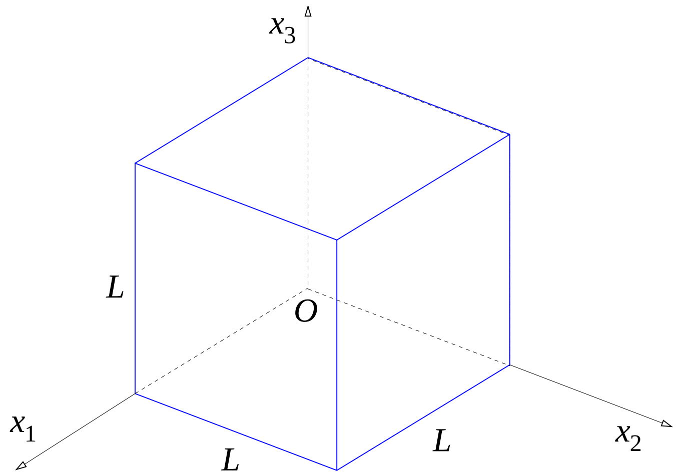

Figure 1: A uniform cube of mass M. Consider rotation about the corner at the origin O.

So denoting the position of an infinitesimal mass dm by $\mathbf{r}$, our moment of inertia tensor is:

$$\mathbf{I} = \int dm \left(r^2 \mathbf{1} - \mathbf{r} \otimes \mathbf{r} \right) \ ,$$

or in components

$$\mathbf{I}_{ij} = \int dm \left((r_k r_k)\delta_{ij} - r_i r_j \right) \ .$$

The volume of the cube is L^3 and therefore the density is $\rho = M/L^3$. So, as $\mathbf{r} = (x_1, x_2, x_3)$, we can write

$$dm = \rho dv = \rho dx_1 dx_2 dx_3 \ ,$$

and integrate each coordinate from 0 to L.

Let us evaluate the nine components of the tensor then. Well, it is symmetric, $\mathbf{I}_{ji} = \mathbf{I}_{ij}$, so we only need evaluate six of them to get them all, right?

Better than that, the computation for all of the diagonal elements are much the same as each other, *e.g.*,

$$\begin{aligned} \mathbf{I}_{11} &= \rho \int dx_1 dx_2 dx_3 (x_1^2 + x_2^2 + x_3^2 - x_1^2) = \rho \int_0^L dx_1 \int_0^L dx_3 \left(\int_0^L dx_2 (x_2^2 + x_3^2) \right) \\ &= \frac{2}{3} \rho L^5 = \frac{2}{3} M L^2 \ . \end{aligned}$$

You should check that the computation for any of $\mathbf{I}_{22}, \mathbf{I}_{33}$ is of the same structure, with the same result. Similarly, the result for any of the off–diagonal components is like this:

$$\mathbf{I}_{12} = \rho \int dx_1 dx_2 dx_3 (-x_1 x_2) = -\frac{1}{4}\rho L^5 = -\frac{1}{4} M L^2 \ .$$

Check that the computations for components $\mathbf{I}_{13}, \mathbf{I}_{23}$ give the same result.

So, in summary, our tensor can be written as a 3×3 symmetric matrix:

$$\mathbf{I} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} M L^2 \ .$$

Now, what do we do with this result now that we have we computed it? Well, unfortunately, (and I mean that), we are not doing a course on motion of rigid bodies in general, otherwise I could show you a lot of really nice stuff, like why tops and frisbees wobble when they spin, and why it is hard to fall over once you bicycle gets going steadily, *etc.*

Suffice it to say that, as we computed when we derived the tensor, it is the thing that comes in when you compute things like angular momentum $\mathbf{L}$, and kinetic energy T, for rotating objects. In general the angular momentum and kinetic energy for rotation about an axis specified by angular velocity $\underline{\omega}$ is:

$$\mathbf{L} = \mathbf{I} \cdot \underline{\omega} \ ; \quad T = \frac{1}{2}\underline{\omega} \cdot \mathbf{L} = \frac{1}{2}\underline{\omega} \cdot \mathbf{I} \cdot \underline{\omega} \ .$$

For example, if the angular velocity $\underline{\omega}$ was entirely about the x–axis, $\underline{\omega} = \omega \mathbf{i} \equiv \omega \mathbf{e}_1$, the angular momentum would be

$$\mathbf{L} = M L^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} M L^2 \omega \ ,$$

which is **not** in the same direction as the angular velocity. This is true in general, as we have seen, and so the question arises as to what are the axes about which the angular momentum would be in the same direction as the angular velocities:

$$\mathbf{L} = I \underline{\omega} \ ,$$

where I would be the simple moment of inertia about the axis that you recall from the good old days of Core A/B. These are the **Principal Axes**, as we discussed in the lectures, and finding them is the eigenvalue problem for the matrix above. There are three solutions, in general.

Principal Axes

Well, I won't bother you with the computation, as I am sure that you can do it easily. The result of finding the eigenvalues is $1/6, 11/12, 11/12$, and this means that two of the eigensolutions are degenerate. The eigenvector corresponding to $1/6$ is $(1,1,1)$. It is nice to choose all of the eigenvectors to orthogonal, and my choice is

$$I_1 = \frac{1}{6}ML^2 \ , \mathbf{n}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \ ;$$
$$I_2 = \frac{11}{12}ML^2 \ , \mathbf{n}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} \ ; \quad I_3 = \frac{11}{12}ML^2 \ , \mathbf{n}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix} \ ,$$

where normalised eigenvectors have been written.

So, a moment playing around will tell you that the solution $I_1, \mathbf{n}_1$ corresponds to an axis running through the body diagonal of the cube, which is an intuitively nice axis, I hope you'll agree (take a cube and try rotating it about that axis....it seems an easy axis, right?!). (See figure 2.) The other two are a pair of orthogonal axes in a plane perpendicular to the above. (Any such pair will do; there is a symmetry...see later.)

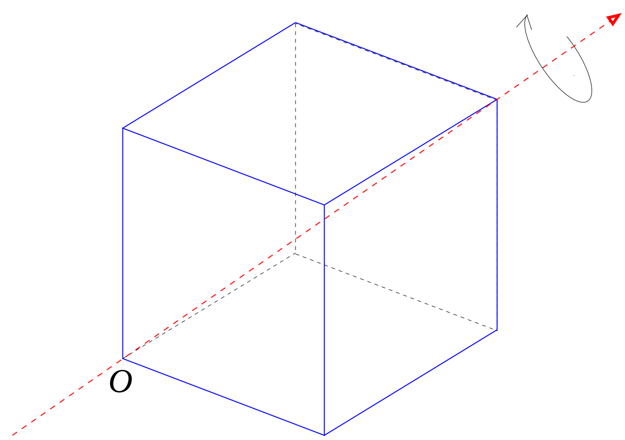

Figure 2: The $(1,1,1)$ Principal Axis.

It makes sense to want to change our definition of the coordinate axes so that they point along those nice directions. Then our moment of inertia tensor with respect to those axes will be $\mathbf{I}'$:

$$\mathbf{I}' = \begin{pmatrix}\frac{1}{6} & 0 & 0\\ 0 & \frac{11}{12} & \\ 0 & 0 & \frac{11}{12}\end{pmatrix} ML^2 \ . \qquad (1)$$

How do we get from $\mathbf{I}$ to $\mathbf{I}'$? Well, that's easy. We know that it must be a rotation, since this is the point of all we have been doing by studying how tensors transform under coordinate transformations, *etc.*!

But you also (might) know that once we know a matrix and its normalised eigenvectors $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$, we can diagonalise it with a matrix M constructed out of them, as follows:

$$M = \begin{pmatrix} \mathbf{n}_1^T \\ \mathbf{n}_2^T \\ \mathbf{n}_3^T \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

We then go from $\mathbf{I}$ to $\mathbf{I}'$, with the equation (called a **Similarity Transformation**):

$$\mathbf{I}' = M\mathbf{I}M^{-1} \ . \tag{2}$$

Try it and see!

What is the relation between this and a rotation which moves the (x_1, x_2, x_3) axes to new axes (x'_1, x'_2, x'_3) with respect to which the moment of inertia tensor is $\mathbf{I}'$? Well, notice that $M^{-1} = M^T$ and that $\det M = +1$, and therefore it is an orthogonal matrix. Look. Our rule for the transformation of the tensor $\mathbf{I}$ under the rotation represented by M is:

$$\mathbf{I}'_{ij} = M_{il}M_{jm}\mathbf{I}_{lm} \ .$$

Rearrange these (since, once we have put indices on them, they are just numbers) and swap the indices on the second M to make it the transpose matrix:

$$\mathbf{I}'_{ij} = M_{il}\mathbf{I}_{lm}M^T_{mj} = M_{il}\mathbf{I}_{lm}M^{-1}_{mj} \ ,$$

(the last step uses orthogonality), and we can write this out without indices to get the similarity transformation we saw before in equation (2).

So for a tensor such as the inertia tensor, (symmetric, rank two), the eigenvalue problem has a very geometrical meaning, that of locating the principal axes and their moments of inertia. Furthermore, diagonalisation does too, meaning that we have performed a rotation of the coordinate axes to align them with the principal axes.

Here, M is the rotation matrix in this case. Actually, it can be written as the product $M = R_2R_1$, where

$$R_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ; \quad R_2 = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \\ 0 & 1 & 0 \\ -\sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \ .$$

These are easily recognisable as special forms of the rotations that we saw when we studied the most general form of a rotation matrix in problem sheet 1 (# 6) and in the tutorials. R_1 rotates by $\cos^{-1}(\frac{1}{\sqrt{2}}) = 45^o$ about the x_3–axis, and so we have new axes (x''_1, x'_2, x_3). R_2 is a rotation about the x'_2 axis by $\cos^{-1}(\sqrt{\frac{2}{3}})$, to put the old x_1 along the body diagonal of the cube, giving us new axes (x'_1, x'_2, x'_3), achieving the diagonalisation of our matrix, and the aligning of the cube along new coordinate axes which coincide with its principal axes.

Notice that the lower 2×2 block of the diagonalised tensor in equation (1) is proportional to the identity, since the eigenvalues are the same. This means that any rotation matrix which acts in that plane will give the same tensor. So we can use any orthogonal pair of axes in that plane as the other principal axes!

Moment of Inertia About the Centre of Mass

Let's compute the moment of inertia tensor about the centre of mass of the cube. Let's call it $\mathbf{I}^{\rm CM}$. We can do this in two ways at least. One is to relocate the origin of the coordinate axes there and then recompute as we did in the previous case.

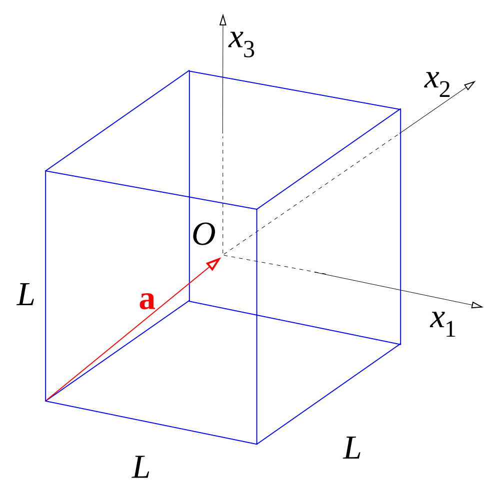

Figure 3: Consider rotation about the centre of mass, shifting the origin of coordinates.

Another way is to use the **Generalised Parallel Axis Theorem**, which we derived in the lectures: If the axes at the centre of mass are separated by a vector $\mathbf{a}$ from the original ones, then the moment of Inertia tensors are related as follows:

$$\mathbf{I} = \mathbf{I}^{\rm CM} + M\left(a^2\mathbf{1} - \mathbf{a}\otimes\mathbf{a}\right) \ .$$

Well, in this case, the vector $\mathbf{a} = (L/2, L/2, L/2)$.

It is easy to see how the components work out. Let's try it. For a diagonal one:

$$\mathbf{I}^{\rm CM}_{11} = \mathbf{I}_{11} - M(a_2^2 + a_3^2) = \frac{2}{3}ML^2 - 2M\frac{L^2}{4} = \frac{1}{6}ML^2 \ .$$

Meanwhile, for an off–diagonal one, it is

$$\mathbf{I}^{\rm CM}_{12} = \mathbf{I}_{12} - M(-a_1a_2) = -\frac{1}{4}ML^2 + M\frac{L^2}{4} = 0 \ .$$

You should check that this is the same for all off–diagonals, and that the previous is the same for all diagonal ones.

So the final result is:

$$\mathbf{I}^{\rm CM} = \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} ML^2 \ .$$

This result is very nice!

First of all, it is diagonal, so this means that the principal axes are the coordinate axes through the centre of mass, pointing perpendicularly through the faces of the cube. These are very intuitive axes, like the body diagonal, as we saw before. See figure 4*(a)*.

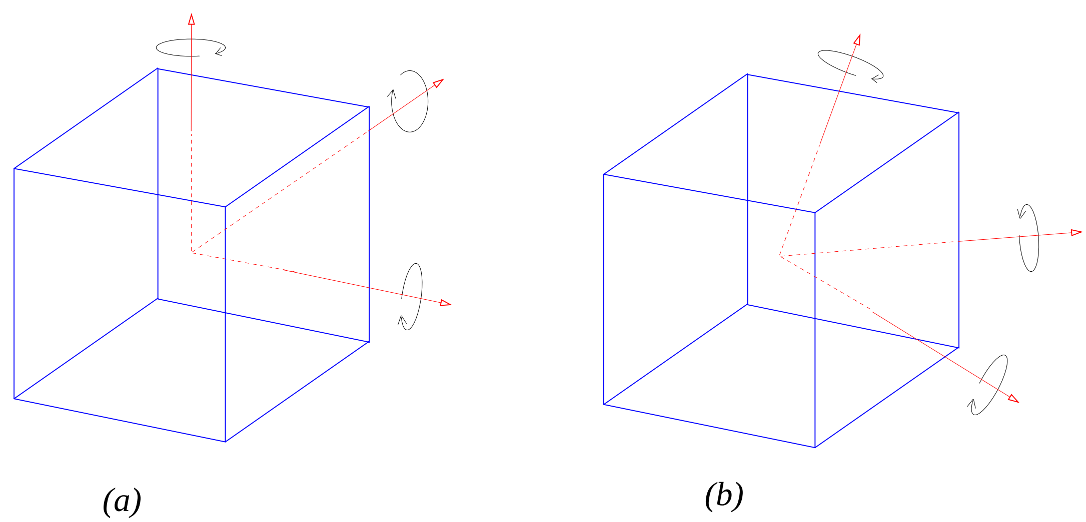

Figure 4: *(a)* Principal axes through the faces. *(b)* Principal axes through **any** other set of axes related by a rotation to those in *(a)*!

There is another excellent property of this result, however. It is proportional to the identity. This means that *any* rotation that we do to new axes, at any angle to these ones, will also have the same moment of inertia tensor. Furthermore, those will be the principal axes too! (See figure 4*(b)*.) So the result is that any set of mutually orthogonal axes through the centre of mass of the cube serve as principal axes for the cube, with identical moments of inertia $I_1 = I_2 = I_3 = ML^2/6$. (Note in particular that the body diagonal we discovered before is one of these!)

This is less intuitively obvious, but follows from the structure that the moment of inertia tensor has. It cannot distinguish between the symmetries of a cube and that of a sphere with the same origin, except that the moments of inertia will be $I_1 = I_2 = I_3 = 2ML^2/5$, where L is the radius of the sphere. (So you could tell it was a sphere you were rotating by doing a measurement, if you knew M and L....of course, you could just look!)

Let's stop here—*cvj*