2H Mathematical Physics — Michaelmas Term

Supplement Sheet 1: Reminder of Moments of Inertia

This is a preparatory review of things that you already know about the "Moment of Inertia" of a rigid body about an axis. The great thing is that in a few steps during Lectures 4 and 5, you will discover that the underlying geometrical object is the "Moment of Inertia Tensor", one of our first main examples of a rank two tensor.

Consider the following figure:

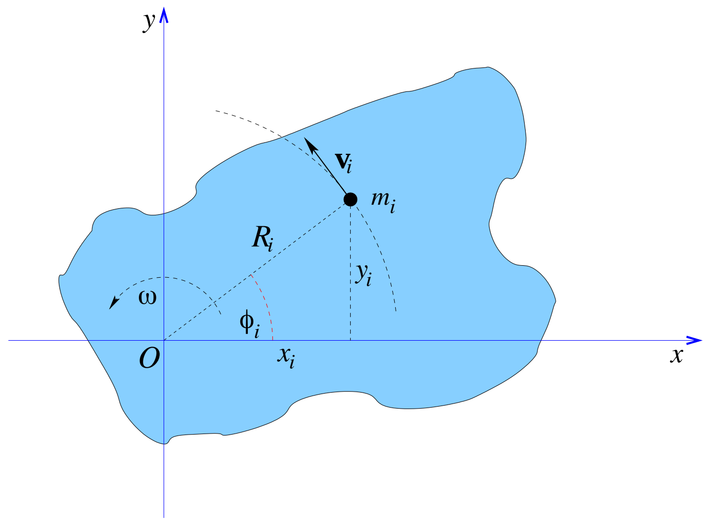

Figure 1: A rigid body rotating about the z–axis with angular velocity ω. Highlighted is one of the many typical particles making up the body. It is a particle which we label i, of mass m_i, a perpendicular distance R_i away.

The path of the particle is a circle of radius $R_i = (x_i^2 + y_i^2)^{1/2}$.

- The speed of the particle is $v_i = R_i\omega$.
- The components of the velocity are (look at figure):

$$\begin{aligned} \dot{x}_i &= -v_i \sin\phi_i = -\omega y_i \\ \dot{y}_i &= v_i \cos\phi_i = \omega x_i \\ \dot{z}_i &= 0 \ , \end{aligned}$$

or, more succinctly:

$$\mathbf{v}_i = \omega \times \mathbf{r}_i \ , \quad \text{where} \quad \omega = \mathbf{k}\omega \ .$$

- The kinetic energy of rotation is

$$T_{\rm rot} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \left(\sum_i m_i R_i^2 \right) \omega^2 = \frac{1}{2} I_z \omega^2 \ .$$

  We have *defined* the **"moment of inertia"** about the z–axis to be

$$I_z = \sum_i m_i R_i^2 = \sum_i m_i \left(x_i^2 + y_i^2 \right)$$

  We'll see it show up in other quantities, for example:

- The Angular momentum's component about the z–axis is

$$\begin{aligned} L_z = \mathbf{k} \cdot \left(\sum_i \mathbf{r}_i \times \mathbf{p}_i \right) &= \mathbf{k} \cdot \left(\sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \right) \\ &= \sum_i m_i \left(x_i \dot{y}_i - y_i \dot{x}_i \right) \\ &= \sum_i m_i (x_i^2 + y_i^2) \omega = \sum_i m_i R_i^2 \omega = I_z \omega \ . \end{aligned} \quad (1)$$

- As usual, we want the sum over all particles to be replaced by a nice integral in a smooth limit:

$$I_z = \sum_i m_i R_i^2 \Longrightarrow I_z = \int R^2 dm \ .$$

Example 1

Imagine a thin uniform rod, or mass m, length a. Lets find the moment of inertia about one end. Look at the figure 2 *(a)*.

The density (mass per unit length) of the rod is $\rho = m/a$. A little bit of rod of length dx is of mass $dm = \rho dx$. So the moment of inertia is

$$I_z = \int_o^a x^2 \rho dx = \frac{1}{3} m a^2 \ .$$

Note that this will also be the result for a uniform square (say) lamina (slab) of side length a. Set up the problem and check it for yourself!

Lets find the moment of inertia about the centre. Look at the figure 2 *(b)*. In this case the moment of inertia is

$$I_z = \int_{-a/2}^{a/2} x^2 \rho dx = \frac{1}{12} m a^2 \ .$$

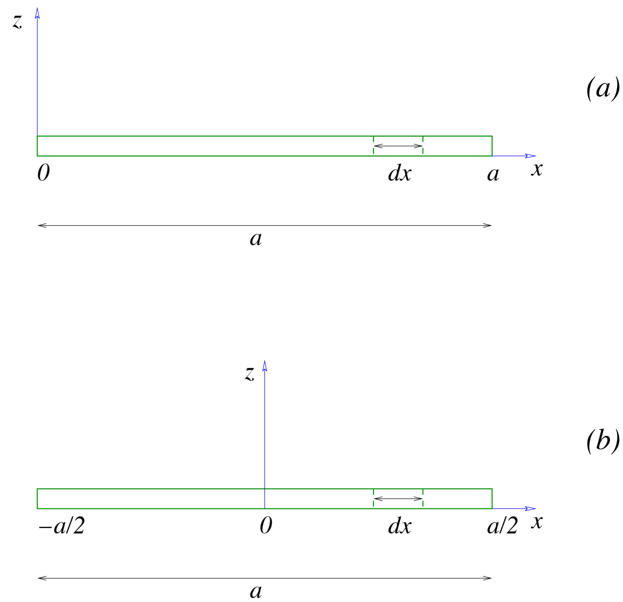

Figure 2: A rigid rod stretched along the x–axis. Consider rotation about *(a)* one end; *(b)* Its centre.

Example 2

Consider a uniform circular disc of radius a and mass m. See the figure 3.

The mass per unit *area* is $\rho = m/(\pi a^2)$. The mass of a circular strip of width dr, and radius r is, to a good approximation: $dm = \rho 2\pi r dr$.

So the moment of inertia is:

$$I_{\text{axis}} = \int_0^a r^2 \rho 2\pi r dr = \frac{1}{2} m a^2 \ .$$

Now recall two useful theorem:

Perpendicular axis theorem

The moment of inertia of any plane lamina about an axis normal to it is equal to the sum of the moments of inertia about any two mutually perpendicular axes passing through the axis of interest, and lying in the plane of the lamina.

So if the lamina is for example lying in the (x, y) plane, then $I_z = I_x + I_y$.

Parallel axis theorem

The moment of inertia of any rigid body about any axis is equal to the moment of inertia about a parallel axis passing through the centre of mass, plus the product of the mass of the body and the square of the distance between the two axes.

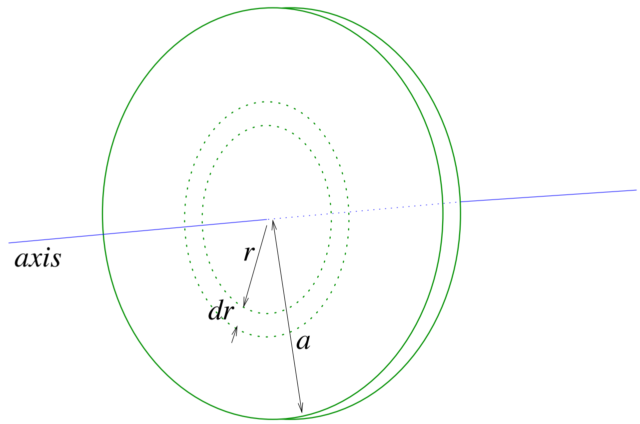

Figure 3: A uniform circular disc of mass m. Consider rotation about its centre.

To check that you remember how these work, show that the moment of inertia of the rod in cases *(a)* and *(b)* above are related by the second theorem. Use the first to show that the moment of inertia of the disc about a diameter is $I = ma^2/4$.