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LETTERS AND COMMENTS

The mass-loaded and nonlinear vibrating string problem revisited

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Abstract. This paper addresses the physics of a vibrating string excited by a hard narrow hammer and the influence of nonlinearities in the time evolution of plucked strings, which have been incorrectly analysed in two recent papers by Bolwell.

In two recently published papers in this journal [1, 2], Bolwell revisits a number of well known and extensively studied problems in the physics of string vibrations. Accepted theoretical models are criticized without justification and '*new solutions*' ([1] abstract) are presented that lead to '*results that are quite frankly astonishing*' ([2] page 318). Unfortunately, both papers are based on incorrect physics so that the results presented are invalid.

In the first paper, 'On Rayleigh's equations for the vibrations of a loaded flexible string', Bolwell first introduces Rayleigh's solutions. These correctly describe the normal modes of string vibration of a string at constant tension T and mass m_0 loaded by a point mass m at a position a along its length L . The normal modes $Y_n(x, t)$ describing the transverse string vibrations are sine waves with nodes at the rigid end-supports of the form $\sin k_n x$ and $\sin k_n(L-x)$ to the left and right of the mass, with $k_n = \omega_n/c$, where $c = \sqrt{TL/m_0}$ is the velocity of transverse waves on the string and we have used the more normal convention for the wavevector k and angular frequency ω. The eigenfrequencies ω_n are determined from the transcendental equation

$$\frac{m}{m_0} \sin k_n a \sin k_n(L-a) = \frac{\sin k_n L}{k_n L}.$$

The important point to note is that the frequencies, and therefore the wavelengths, are perturbed from those of the unloaded string, apart from those modes with a node at the point of mass attachment. For all other modes, the separate sine-wave solutions to the right and the left of the string result in a discontinuity in slope $\Delta(\partial y/\partial x)$ at the point of mass attachment, as required to satisfy the equation of motion,

$$m\frac{\partial^2 y}{\partial t^2} = T\Delta\left(\frac{\partial y}{\partial x}\right).$$

It follows therefore that, since any possible motion of the string can be described in terms of the normal modes of the system, we can write

$$y(x, t) = \sum_n Y_n(x)(A_n \sin\omega_n t + B_n \cos\omega_n t) \tag{1}$$

where the coefficients A_n and B_n are determined by the initial displacements and velocities along the length of the string at $t = 0$. All this is standard textbook material [3, 4].

However, Bolwell then proposes without justification that 'it is especially beneficial to reduce the (above) Rayleigh equation to a single standing-wave solution' of the form

$$y(x, t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)(A_n \sin\omega_n t + B_n \cos\omega_n t) \tag{2}$$

as in equation (8) of [1], where Bolwell has assumed Rayleigh's properly evaluated eigenfrequencies ω_n for the normal modes, but has replaced the proper spatial eigensolutions $Y_n(x)$ by the sine-wave eigenfunctions *for the unloaded string*. Because the sine waves fail to satisfy the boundary condition at a, which requires a discontinuity in slope to accelerate the attached mass, Bolwell's postulated 'alternative approach' is therefore unphysical, quite apart from lacking any mathematical validity. Consequently, all that follows in the paper based on this incorrect model is wrong.

Bolwell then goes on to address the interesting problem of a string excited at a point along its length by a hard hammer, a problem related to the sound produced by a piano or zither. Unfortunately, the problem incorrectly assumes 'a naive equation of motion', which the reader is expected to derive from equation (27) of [1]. It is indeed easy to derive the proposed model, which assumes that the mass is connected to the rigid end-supports by straight-line sections of string. However, it is also easy to show that such a solution is unphysical. Since the net force produced by the tension on any straight section of string is zero, any straight section of a string must therefore be either at rest or moving with uniform velocity. This implies that the mass attached to the string must itself be moving with a constant velocity, which reverses in direction every half cycle, which is again unphysical.

The essential point to recognize is that any disturbance created by the impacting mass travels out towards the end-supports with a finite velocity c. There will therefore always be a time lag in any resulting disturbance travelling along the string; this is the basis of the unwisely criticized approaches taken by '*impulse modellers*'. Only in the limit of a very heavy mass impacting and remaining stuck on the string at very long times, when all modes other than the lowest eigenmode will have decayed by the damping inevitably present, would the solution proposed by Bolwell even approximate to reality.

A proper theoretical description of this problem has been given by Hall [5, 6] for both hard and soft impacting masses. On initial impact, the impacting mass initiates an outgoing wave, which travels outwards in both directions with velocity c towards the end-supports. This wave is then reflected at the ends of the string and provides a renewed impulse to the mass, producing secondary discontinuities in velocity of the attached mass and the generation of a new pair of secondary radiated waves. On reflection from the ends, these secondary waves result in further reflections, further discontinuities in velocity of the attached mass and a new set of secondary waves, which in turn generate further generations of waves. Depending on the magnitude of the impacting mass and its position along the string, a rich variety of solutions is possible. Only for a very light mass impacting close to an end-support will the first reflected wave cause the mass to bounce off the string, provided it has not become permanently attached [5]. In general, many reflections are necessary before the mass bounces back off the string. Indeed, having first bounced off, the string can often make renewed contact with a heavy mass three or four times before contact is permanently lost, as shown by Hall [6].

In the early 1900s, many attempts were made to tackle this problem, but most publications, like Bolwell's, were based on incorrect physical models. In [6] Hall provides a penetrating analysis of the historical work and introduces an elegant and relatively simple way of solving the problem, based on the above model of reflections and propagation of secondary waves. Such a model can be used to describe the complicated dynamics over a wide range of experimental parameters.

As Bolwell correctly suggests, the problem of a mass impacting and bouncing off a string, or the equivalent problem of a trapeze artist bouncing on a 'high wire', is ideally suited to student project work, either experimental or computational. But such projects should be based on sound physical models, such as those described by Hall, rather than on the incorrect theoretical model presented in [1].

In the second paper [2], Bolwell asks 'How realistic is the D'Alembert plucked string?'. He addresses the influence of finite-amplitude string vibrations and the resulting second-order corrections to the standard linear wave equation. This problem is properly described in advanced textbooks on vibrations, such as Morse and Ingard [3]. A finite-amplitude string vibration $y(x, t)$ leads to an increase in the total length of the string

$$\delta L = \frac{1}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x$$

which results in the well known expression for the potential energy of the string $T\delta L$, essentially the work done in stretching the string assuming no change in tension. However, the increase in stretched length associated with a finite-amplitude standing wave, $y(x, t) = a \sin(n\pi x/L) \sin \omega_n t$, will result in a time-varying tension. To second order in the amplitude of string vibration this tension can be written in the useful form

$$T\left[1 + \frac{n^2\pi^2}{8}\frac{a^2}{L\Delta L}(1 - \cos 2\omega_n t)\right] \quad (3)$$

where ΔL is the amount by which the string was initially stretched to achieve the tension T. In deriving this expression, we have assumed that local increases in string length result in an immediate increase in string tension, which remains uniform along its length. This is equivalent to assuming that the velocity of longitudinal waves on the string is very much larger than that of transverse waves and that we can ignore the dynamics of the longitudinal modes, which is valid for most examples of experimental interest. Morse and Ingard [3] present a more general mathematical model, which allows for a finite coupling between the transverse and longitudinal modes. However, for cases of practical interest, the above simplifying assumption is sufficient.

The above expression shows that finite-amplitude string vibrations result in a net increase in average tension and a time-varying term at double the frequency of the mode excited. For example, a string plucked at its centre involves only the odd eigenmodes with frequencies, $f_1, 3f_1, 5f_1, \ldots$ etc. However, the nonlinear frequency-doubling term will transfer energy to the supporting structure at a frequency $2f_1$, leading to a component at this frequency in the sound excited, as demonstrated by Legge and Fletcher in [7] figure 3.

The most important effect of the nonlinearity, however, is to increase the mean tension, which in turn leads to a fractional increase in the transverse wave velocity c, and hence frequency, of any excited mode by an amount $\Delta f_n/f_n = \frac{1}{2}\Delta T/T$. Contrary to what is stated by Bolwell, this is a rather small effect for most stringed instruments, otherwise the pitch of a note would change appreciably when it is played loudly. For example, consider a string of length 30 cm, typically stretched by about 3 cm to provide its normal playing tension. Assume that it is bowed or plucked strongly to give a very large vibration amplitude of 1 cm at the centre of the string. The fractional increase in the fundamental frequency expected from equation (3) is $\sim a^2/(2\Delta L\, L) = 1/600$, which would represent an imperceptible increase in pitch. To investigate the very interesting nonlinear physics of vibrating strings [8], it is advantageous to use a very slack string, so that not only is ΔL small but very-large-amplitude vibrations can also be excited.

Although the above expression for the nonlinear increase in tension disagrees by a numerical factor from that given by Bolwell, he correctly predicts an increase in tension with mode number varying as n^2. However, an incorrect assumption is then made that each mode acts independently, so that 'the value c_1 (i.e. the transverse wave velocity) is different for each standing wave', where the term in parentheses is my own. This is quite unphysical. Any increase in tension clearly affects all excited modes alike. It is therefore not surprising that results derived based on this model are 'quite frankly astonishing'. Unfortunately, the results are simply wrong.

A more appropriate approach would be to evaluate the increase in tension from all the modes present. Because the normal modes are orthogonal, the increase in string length is simply given by

$$\frac{1}{2}\sum a_n^2 \frac{n^2\pi^2}{L}$$

summed over all excited modes. This leads to the frequency of all the modes being increased by the same fractional amount, proportional to the mean square amplitude of transverse string displacement. For a string released from rest by plucking at its centre, the example used by Bolwell, the amplitudes of the various modes initially present are such that $a_n \sim 1/n^2$, with only the odd modes of string vibration excited. The nonlinear effects will therefore be dominated by the fundamental mode, contributing a term nine times larger than the third harmonic and 25 times larger than the fifth.

Throughout the two papers, Bolwell claims major deficiencies in standard textbook treatments of waves on strings and is particularly critical of mathematical, and supposedly unphysical, approaches used by ‘impulse modellers’, which are based on the D’Alembert solutions of the wave equation. Nothing in either paper supports such a view. Readers interested in learning more about the interesting physics of string vibrations and about the real factors that complicate their vibrations on musical instruments are strongly recommended to read the standard textbooks cited by Bolwell. Morse and Ingard [3] provide an authoritative theoretical account of both linear and nonlinear string vibrations, while Fletcher and Rossing [4] describe relevant theory in relation to applications in musical acoustics. The latter can be strongly recommended as an invaluable source of ideas for interesting computational and experimental student projects involving string vibrations and much else in musical acoustics.

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